ECON30401: Time Series Econometrics

Lecture 2: ARMA(p.q) Processes & Stationarity

Nicky L. Grant

October 2016

Table of contents

Overview

Recap of Last Lecture

Time Series Process vs. Realisation

Defined Stationarity & White Noise

Introduced AR(1),MA(q),ARMA(1,1) Processes

Derived ACF of MA(q)

Today's Lecture

Introduce MA(∞) Process & its importance

Derive Properties of AR(1)

Outline properties of ARMA(1,1) [Derivations in Exercise 2]

Introduce the Lag Operator and its use

Introduce ARMA(p,q) Models

Conditions for Stationarity in ARMA(p,q)

MA(∞) Process

MA(∞) Process

Definition: MA(q) Process

$$Y_t = \alpha + \sum_{j=0}^{q} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \mathrm{WN}(\sigma^2)$$

MA(∞) Process

Definition: MA(q) Process

$$Y_t = \alpha + \sum_{j=0}^{q} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \mathrm{WN}(\sigma^2)$$

Definition: MA(∞) Process

$$Y_t = \alpha + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \mathrm{WN}(\sigma^2)$$

MA(∞) Process

Definition: MA(q) Process

$$Y_t = \alpha + \sum_{j=0}^{q} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \mathrm{WN}(\sigma^2)$$

Definition: MA(∞) Process

$$Y_t = \alpha + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \mathrm{WN}(\sigma^2)$$

$$\mathbb{E}\left[Y_t\right] = \alpha$$

MA(∞) Process

Definition: MA(q) Process

$$Y_t = \alpha + \sum_{j=0}^{q} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \mathrm{WN}(\sigma^2)$$

$$\mathbb{E}\left[Y_t\right] = \alpha$$

Definition: MA(∞) Process

$$Y_t = \alpha + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \mathrm{WN}(\sigma^2)$$

$$\mathbb{E}\left[Y_t\right] = \alpha$$

MA(∞) Process

Definition: MA(q) Process

$$Y_t = \alpha + \sum_{j=0}^{q} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \mathrm{WN}(\sigma^2)$$

Definition: MA(∞) Process

$$Y_t = \alpha + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \mathrm{WN}(\sigma^2)$$

$$\mathbb{E}[Y_t] = \alpha$$

$$\mathbb{E}[Y_t] = \alpha$$

$$Var[Y_t] = \sigma^2 \sum_{j=0}^{q} \theta_j^2$$

MA(∞) Process

Definition: MA(q) Process

$$Y_t = \alpha + \sum_{j=0}^{q} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \mathrm{WN}(\sigma^2)$$

Definition: MA(∞) Process

$$Y_t = \alpha + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \mathrm{WN}(\sigma^2)$$

$$\mathbb{E}[Y_t] = \alpha$$

$$\mathbb{E}[Y_t] = \alpha$$

$$Var[Y_t] = \sigma^2 \sum_{j=0}^{q} \theta_j^2$$

$$Var[Y_t] = \sigma^2 \sum_{j=0}^{\infty} \theta_j^2$$

MA(∞) Process

Definition: MA(q) Process

$$Y_t = \alpha + \sum_{j=0}^{q} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \mathrm{WN}(\sigma^2)$$

Definition: MA(∞) Process

$$Y_t = \alpha + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \mathrm{WN}(\sigma^2)$$

$$\mathbb{E}[Y_t] = \alpha$$

$$\mathbb{E}[Y_t] = \alpha$$

$$Var[Y_t] = \sigma^2 \sum_{j=0}^{q} \theta_j^2$$

$$Var[Y_t] = \sigma^2 \sum_{j=0}^{\infty} \theta_j^2$$

$$\gamma(k) = \begin{cases} \sigma^2 \sum_{s=0}^{q-k} \theta_s \theta_{s+k} & : k = 0, ., q \\ 0 & : k > q \end{cases}$$

MA(∞) Process

Definition: MA(q) Process

$$Y_t = \alpha + \sum_{j=0}^{q} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \text{WN}(\sigma^2)$$

$$\mathbb{E}[Y_t] = \alpha$$

$$Var[Y_t] = \sigma^2 \sum_{j=0}^{q} \theta_j^2$$

$$\gamma(k) = \begin{cases} \sigma^2 \sum_{s=0}^{q-k} \theta_s \theta_{s+k} & : k = 0, ., q \\ 0 & : k > q \end{cases}$$

Definition: MA(∞) Process

$$Y_t = \alpha + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} \quad \varepsilon_t \sim \text{WN}(\sigma^2)$$

$$\mathbb{E}[Y_t] = \alpha$$

$$Var[Y_t] = \sigma^2 \sum_{j=0}^{\infty} \theta_j^2$$

$$\gamma(k) = \sigma^2 \sum_{s=0}^{\infty} \theta_s \theta_{s+k} \qquad k = 0, 1, \ldots$$

Condition for Stationarity of MA(∞)

Y_t is MA(∞) then $\mathbb{E}[Y_t]$, $Var[Y_t]$, $Cov(Y_t, Y_{t-j})$ not a function of t.

If $Var[Y_t] < \infty$ then MA(∞) satisfies Stationarity Condition.

Definition: Squared Summability(SS)

$$\sum_{s=0}^{\infty} \theta_s^2 < \infty$$

Definition: Absolute Summability(AS)

$$\sum_{s=0}^{\infty} |\theta_s| < \infty$$

Stationary if MA(∞) coefficients Absolutely Summable

Importance of MA(∞): Wold Decomposition

Theorem 2.1: Wold Decomposition

Any stationary process can be represented as $MA(\infty)$:

$$Y_t = \alpha + \sum_{s=0}^{\infty} \theta_s \varepsilon_{t-s}, \quad \theta_0 = 1 \qquad \varepsilon_t \sim WN(\sigma^2)$$

for $\theta_1, .., \theta_\infty$ satisfying Square Summability

Derive mean, variance and ACF of any stationary process by finding its MA(∞) form and using general formulas for MA(∞).

We use this method for the stationary AR(1) Process

AR(1) Process

AR(1) Process $|\phi_1| < 1$

$$
\begin{aligned}
Y_t &= \mu + \phi_1 Y_{t-1} + \varepsilon_t \\
&= \mu + \mu\phi_1 + \phi_1^2 Y_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \qquad \text{(Subst. } Y_{t-1} = \mu + \phi_1 Y_{t-2} + \varepsilon_{t-1}\text{)} \\
&= \mu + \mu\phi_1 + \mu\phi_1^2 + \phi_1^3 Y_{t-3} + \phi_1^2 \varepsilon_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \qquad \text{(Subs. } Y_{t-2} = \mu + \phi_1 Y_{t-3} + \varepsilon_{t-2}\text{)} \\
&\vdots \\
&= \mu + \mu\phi_1 + .. + \mu\phi_1^j + \phi_1^{j+1} Y_{t-(j+1)} + \phi_1^j \varepsilon_{t-j} + \phi_1^{j-1} \varepsilon_{t-j+1} + ... + \varepsilon_t \\
&\vdots \\
&= \frac{\mu}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j \varepsilon_{t-j} \qquad \text{if } \ |\phi_1| < 1 \quad \phi_1^{j+1} Y_{t-(j+1)} \to 0 \text{ as } j \to \infty
\end{aligned}
$$

MA(∞) Process where $\alpha = \frac{\mu}{1-\phi_1}$ and $\theta_s = \phi_1^s$

AR(1): Mean & Variance

MA(∞) form of stationary AR(1): $Y_t = \frac{\mu}{1-\phi_1} + \sum_{s=0}^{\infty} \phi_1^s \varepsilon_{t-s}$

Mean

$\mathbb{E}[Y_t] = \frac{\mu}{1-\phi_1}$

Variance

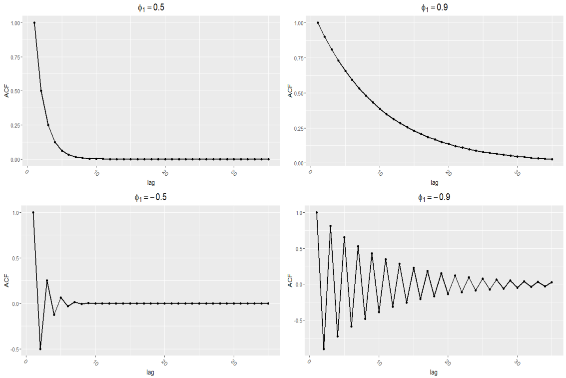
$$
\begin{aligned}
Var(Y_t) &= \sigma^2 \sum_{s=0}^{\infty} \theta_s^2 \\
&= \sigma^2 \sum_{s=0}^{\infty} (\phi_1^s)^2 \\
&= \sigma^2 (1 + \phi_1^2 + \phi_1^4 + ...) \\
&= \sigma^2 \frac{1}{1-\phi_1^2} \qquad \text{if } |\phi_1| < 1
\end{aligned}
$$

AR(1): Autocovariance and Autocorrelation Function

Autocovariance Function

$$
\begin{aligned}
\gamma(k) &= \sigma^2 \sum_{s=0}^{\infty} \theta_s \theta_{s+k} \\
&= \sigma^2 \sum_{s=0}^{\infty} \phi_1^s \phi_1^{s+k} \\
&= \phi_1^k \sigma^2 \sum_{s=0}^{\infty} \phi_1^{2s} \\
&= \phi_1^k \gamma(0) \qquad k = 0, 1, 2, \ldots
\end{aligned}
$$

Autocorrelation Function $\rho(k) = \phi_1^k$



ARMA(1,1)

MA(∞) form of ARMA(1,1)

$$
\begin{aligned}
Y_t &= \mu + \phi_1 Y_{t-1} + \eta_1 \varepsilon_{t-1} + \varepsilon_t \\
&= \mu + \mu\phi_1 + \phi_1^2 Y_{t-2} + \phi_1 \eta_1 \varepsilon_{t-2} + (\phi_1 + \eta_1)\varepsilon_{t-1} + \varepsilon_t \\
&= \mu + \mu\phi_1 + \mu\phi_1^2 + \phi_1^3 Y_{t-3} + \phi_1^2 \eta_1 \varepsilon_{t-3} + \phi_1(\phi_1 + \eta_1)\varepsilon_{t-2} + (\phi_1 + \eta_1)\varepsilon_{t-1} + \varepsilon_t \\
&\vdots \\
&= \mu + \mu\phi_1 + .. + \mu\phi_1^j + \underline{\phi_1^{j+1} Y_{t-j-1} + \eta_1 \phi_1^j \varepsilon_{t-j-1}} + \phi_1^{j-1}(\eta_1 + \phi_1)\varepsilon_{t-j} + .. + (\eta_1 + \phi_1)\varepsilon_{t-1} + \varepsilon_t \\
&\vdots \\
&= \frac{\mu}{1-\phi_1} + (\eta_1 + \phi_1) \sum_{j=1}^{\infty} \phi_1^{j-1} \varepsilon_{t-j} + \varepsilon_t
\end{aligned}
$$

ARMA(1,1) stationary when $|\phi_1| < 1$

ARMA(1,1) Properties [Derived in Video Exercise 2]

$$\mathbb{E}[Y_t] = \frac{\mu}{1-\phi_1}$$

$$Var[Y_t] = \sigma^2 \frac{(1+2\phi_1\eta_1+\eta_1^2)}{1-\phi_1^2}$$

$$\gamma(k) = \begin{cases} \sigma^2(\phi_1+\eta_1)\left[1+\frac{(\phi_1+\eta_1)}{1-\phi_1^2}\right] & : k=1 \\ \phi_1^{k-1}\gamma(1) & : k>1 \end{cases}$$

ARMA(p,q) Process

ARMA(p,q) Process

Definition: Autoregressive Process of Order p (AR(p))

$$Y_t = \mu + \phi_1 Y_{t-1} + ... + \phi_p Y_{t-p} + \varepsilon_t \quad \varepsilon_t \sim WN(\sigma^2)$$

Definition: Autoregressive Moving Average Process order (p,q)

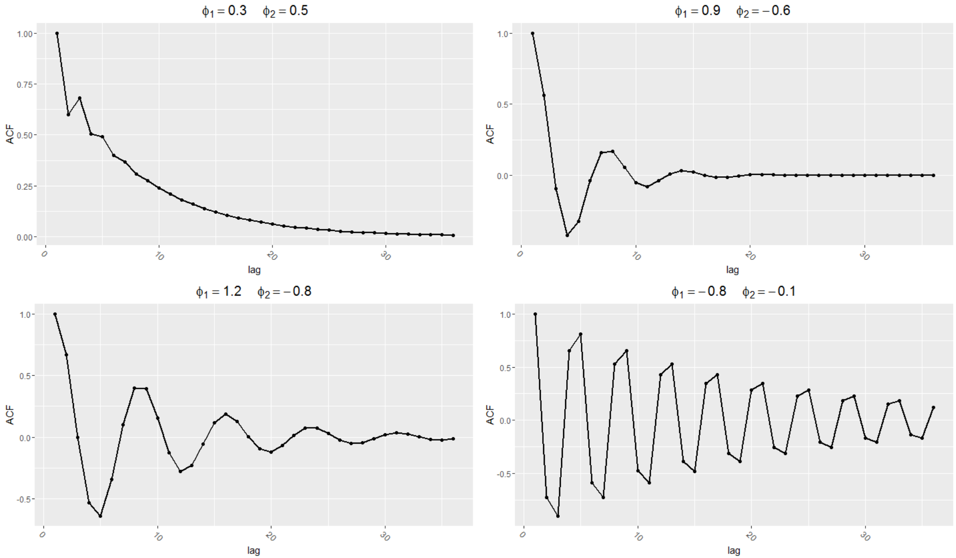
$$Y_t = \mu + \phi_1 Y_{t-1} + ... + \phi_p Y_{t-p} + \varepsilon_t + \eta_1 \varepsilon_{t-1} + ... + \eta_q \varepsilon_{t-q} \quad \varepsilon_t \sim WN(\sigma^2)$$

$p = 0$ is MA(q)

$q = 0$ is AR(p)

ARMA(p,q) general class of processes capable of modelling complex dynamics

Example ACF of AR(2)



Allows more complex correlation patterns relative to AR(1)

Stationarity ARMA(p,q) Processes

Useful Tool: Lag Operator

Definition: Lag Operator

The Lag Operator L satisfies

$$LY_t = Y_{t-1} \qquad \text{for any t}$$

Useful Tool: Lag Operator

Definition: Lag Operator

The Lag Operator L satisfies

$$LY_t = Y_{t-1} \qquad \text{for any t}$$

Important Property: L 'behaves' like Polynomial

$$\begin{aligned}(1-L)(1+L)Y_t &= (1-L)(Y_t + LY_t)\\ &= (1-L)(Y_t + Y_{t-1})\\ &= (Y_t + Y_{t-1})) - L(Y_t + Y_{t-1})\\ &= (Y_t + Y_{t-1}) - Y_{t-1} - Y_{t-2}\\ &= Y_{t-1} - Y_{t-2}\end{aligned}$$

$$\begin{aligned}(1-L^2)Y_t &= Y_t - L^2 Y_t\\ &= Y_t - Y_{t-2}\end{aligned}$$

Intuition for Stationarity Condition in AR(p) for p=2

$$Y_t = \mu + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \quad \Leftrightarrow \quad (1 - \phi_1 L - \phi_2 L^2) Y_t = \mu + \varepsilon_t$$

$$(1 - \phi_1 L - \phi_2 L^2) Y_t = (1 - \lambda_1 L)(1 - \lambda_2 L) Y_t$$

Intuition for Stationarity Condition in AR(p) for p=2

$$Y_t = \mu + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \quad \Leftrightarrow \quad (1 - \phi_1 L - \phi_2 L^2) Y_t = \mu + \varepsilon_t$$

$$(1 - \phi_1 L - \phi_2 L^2) Y_t = (1 - \lambda_1 L)(1 - \lambda_2 L) Y_t$$

$$Y_t = (1 - \lambda_1 L)^{-1} (1 - \lambda_2 L)^{-1} (\mu + \varepsilon_t)$$

Require $|\lambda_1| < 1$ and $|\lambda_2| < 1$ for stationarity

Suppose not and $\lambda_2 = 1$

$$\begin{aligned} (1 - \lambda_1 L) Y_t &= (1 - L)^{-1} (\mu + \varepsilon_t) \\ &= (1 + L + L^2 + \ldots)(\mu + \varepsilon_t) \\ &\rightarrow \infty \end{aligned}$$

Stationarity Condition in general AR(p)

$$Y_t = \mu + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \varepsilon_t \quad \Leftrightarrow \quad (1 - \phi_1 L - \cdots \phi_p L^p) Y_t = \alpha + \varepsilon_t$$

$$(1 - \phi_1 L - \cdots - \phi_p L^p) Y_t = (1 - \lambda_1 L) \times \cdots \times (1 - \lambda_p) Y_t$$

$$Y_t = (1 - \lambda_1 L)^{-1} \times \cdots \times (1 - \lambda_p L)^{-1} (\mu + \varepsilon_t)$$

$|\lambda_j| < 1$ for $j = \{1, .., p\}$ for Stationarity

Lag Operator & AR(p) Stationarity: Extra video material

Stationarity Condition for ARMA(p,q) Processes

$$Y_t = \mu + \phi_1 Y_{t-1} + ... + \phi_p Y_{t-p} + \varepsilon_t + \eta_1 \varepsilon_{t-1} + ... + \eta_q \varepsilon_{t-q}$$

$$(1 - \phi_1 L - \cdots \phi_p L^p) Y_t = \mu + \varepsilon_t + \eta_1 \varepsilon_{t-1} + ... + \eta_q \varepsilon_{t-q}$$

$MA(q)$ always stationary

Stationarity of $ARMA$ depends only on AR

Stationarity Condition for ARMA(p,q) same as for $AR(p)$

Recap & Moving on..

Recap

Discussed MA(∞) and its use

Derived properties of AR(1)

Conditions for stationarity of ARMA(p,q) models

Next Week

Sample autocorrelation function

Estimation of ARMA models

Model Selection & hypothesis testing

Model Specification (testing for serial correlation)

Reading: **Chapter 2** of notes and **Chapter 3** in prep for next lecture on Estimation.