Time Series Econometrics

Nicky Grant

ECON5221: Problem Set 1

Semester 2 2019/2020

Exercise Questions 1

The questions in Problem Set 1 are to be covered in Tutorial 1 (as many as we can get through in an hour). The questions in Video Problem Set 1 are to be completed by you also, a full video solution will be given after the first tutorial.

Problem Set 1

In the following questions each process evolves over time $\mathcal{T} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.

1. An $MA(2)$ process is

$$Y_t = \alpha + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} \qquad \varepsilon_t \sim WN(\sigma^2)$$

   (a) Show that

$$\mathbb{E}[Y_t] = \alpha$$

   and

$$Var[Y_t] = \sigma^2(1 + \theta_1^2 + \theta_2^2).$$

   (b) Find the autocorrelation function of Y_t. These autocorrelations should be written in terms of the MA coefficients (θ_1 and θ_2).

2. Consider the AR(2) process

$$Y_t = \mu + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \qquad \varepsilon_t \sim WN(\sigma^2) \tag{1}$$

   (a) Under what conditions on ϕ_1, ϕ_2 is the AR(2) process in (1) weakly stationary.

   (b) Assuming conditions in 2(a) hold, derive the mean of Y_t.

   (c) Assuming conditions in 2(a) holds, derive a second order difference equation for the autocovariance function of Y_t.

   (d) Using 2(c) derive the variance of Y_t and the correlation of Y_t and Y_{t-1}.

3. Consider an i.i.d process $\{Y_t\}$ where Y_t is Cauchy distributed with density function $f_{Y_t}(y) = 1/\pi(1 + y^2)$ for y on the real line $\mathbb{R}$. Show that the process is strictly stationary. Is the process also weakly stationary? [Provide a formal justification for your answer.]

4. If a process $\{Y_T\}$ is weakly stationary with absolutely summable covariances which conditions on its first and second moments does it satisfy? Prove that $\bar{Y}_T = \frac{1}{T}\sum_{t=1}^{T} Y_t$ is a consistent estimator of the mean of Y_t under these conditions. [*Hint: Use Chebychev's inequality*.]

Video Problem Set 1

1. Consider an $MA(\infty)$ process

$$Y_t = \mu + \sum_{s=0}^{\infty} \eta_s \varepsilon_{t-s}$$

where the MA coefficients satisfy the difference equation

$$\eta_s = \phi_1 \eta_{s-1}, \quad s > 1$$

with starting values

$$\eta_0 = 1, \quad \eta_1 = \phi_1 + \theta_1.$$

(a) Obtain $Y_t - \phi_1 Y_{t-1}$ and hence show that the process with this $MA(\infty)$ representation is the $ARMA(1,1)$

$$Y_t = \alpha + \phi_1 Y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

with $\alpha = (1 - \phi_1)\mu$.

(b) Using the difference equation for the $MA(\infty)$ coefficients, show that

$$\eta_s = \phi_1^{s-1} \eta_1, \quad s > 1$$

with $\eta_1 = \phi_1 + \theta_1$.

(c) The absolute summablity condition states that this process is stationary provided

$$\sum_{s=0}^{\infty} |\eta_s| < \infty.$$

Using the result from (b), show that the $ARMA(1,1)$ process is stationary when

$$|\phi_1| < 1.$$

[Hint: Note that for any two values a and b, $|ab| = |a| \times |b|$.]

(d) Again using the result from (b), together with the general result that the variance of the $MA(\infty)$ process is given by

$$Var[Y_t] = \sigma^2 \sum_{s=0}^{\infty} \eta_s^2,$$

show that the variance of the $ARMA(1,1)$ process is

$$\begin{aligned} Var[Y_t] &= \sigma^2 \left\{ 1 + (\phi_1 + \theta_1)^2 \sum_{s=0}^{\infty} \phi_1^{2s} \right\} \\ &= \sigma^2 \frac{(1 + 2\phi_1\theta_1 + \theta_1^2)}{1 - \phi_1^2}. \end{aligned}$$

(e) Show that

$$\gamma(k) = \begin{cases} \sigma^2(\phi_1 + \theta_1)\left[1 + \frac{(\phi_1+\theta_1)\phi_1}{1-\phi_1^2}\right] & : k = 1 \\ \phi_1^{k-1}\gamma(1) & : k > 1 \end{cases}$$

$$\rho(k) = \begin{cases} \frac{(\phi_1+\theta_1)(1+\phi_1\theta_1)}{1+2\phi_1\theta_1+\theta_1^2} & : k = 1 \\ \phi_1^{k-1}\rho(1) & : k > 1 \end{cases}$$

(f) What happens to the $MA(\infty)$ coefficients η_s $(s = 1, 2, ...)$ given above in the special case where $\theta_1 = -\phi_1$. Comment on the result.