Mathematical Economics|EC3304

Multivariate Calculus II

Nicky Grant (Semester 2, 2024/2025)

Table of Contents

Implicit Function Theorem

Motivation: Implicit Functions

For a function $f(y, x) = 0$ then (generally) y and x implictly related

Motivation: Implicit Functions

For a function $f(y, x) = 0$ then (generally) y and x implictly related

When can we find a function $y(x)$ where $f(y(x), x) = 0$

Motivation: Implicit Functions

For a function $f(y, x) = 0$ then (generally) y and x implictly related

When can we find a function $y(x)$ where $f(y(x), x) = 0$

Is $y(x)$ differentiable and continuous in x

Motivation: Implicit Functions

For a function $f(y, x) = 0$ then (generally) y and x implictly related

When can we find a function $y(x)$ where $f(y(x), x) = 0$

Is $y(x)$ differentiable and continuous in x

Implicit function theorem gives conditions when so

Implicit Function Theorem (no proof)

Let f(y,x) be a C^1 function on a ball about (y_0, x_0) in $\mathbb{R}^2$ where $f(y_0, x_0) = c$ and $\frac{\partial f(y_0, x_0)}{\partial y} \neq 0$ then there exists a C^1 function $y = y(x)$ defined on an interval I about the point x_0 such that

Implicit Function Theorem (no proof)

Let f(y,x) be a C^1 function on a ball about (y_0, x_0) in $\mathbb{R}^2$ where $f(y_0, x_0) = c$ and $\dfrac{\partial f(y_0, x_0)}{\partial y} \neq 0$ then there exists a C^1 function $y = y(x)$ defined on an interval I about the point x_0 such that

$$f(y(x), x) = c \text{ for all } y \in I$$

Implicit Function Theorem (no proof)

Let f(y,x) be a C^1 function on a ball about (y_0, x_0) in $\mathbb{R}^2$ where $f(y_0, x_0) = c$ and $\dfrac{\partial f(y_0, x_0)}{\partial y} \neq 0$ then there exists a C^1 function $y = y(x)$ defined on an interval I about the point x_0 such that

$$f(y(x), x) = c \text{ for all } y \in I$$

$$y(x_0) = y_0$$

Implicit Function Theorem (no proof)

Let f(y,x) be a C^1 function on a ball about (y_0, x_0) in $\mathbb{R}^2$ where $f(y_0, x_0) = c$ and $\dfrac{\partial f(y_0, x_0)}{\partial y} \neq 0$ then there exists a C^1 function $y = y(x)$ defined on an interval I about the point x_0 such that

$$f(y(x), x) = c \text{ for all } y \in I$$

$$y(x_0) = y_0$$

$$y'(x_0) = -\frac{\frac{\partial f(y_0,x_0)}{\partial x}}{\frac{\partial f(y_0,x_0)}{\partial y}}$$

Implicit Function Theorem (no proof)

Let f(y,x) be a C^1 function on a ball about (y_0, x_0) in $\mathbb{R}^2$ where $f(y_0, x_0) = c$ and $\frac{\partial f(y_0, x_0)}{\partial y} \neq 0$ then there exists a C^1 function $y = y(x)$ defined on an interval I about the point x_0 such that

$$f(y(x), x) = c \text{ for all } y \in I$$

$$y(x_0) = y_0$$

$$y'(x_0) = -\frac{\frac{\partial f(y_0, x_0)}{\partial x}}{\frac{\partial f(y_0, x_0)}{\partial y}}$$

See SB Theorem 15.3 (pp. 337-340)

Example: Implicit Function Theorem

Marginal rate of techincal substitution between k and l for

$$f(k, l) := 20k^{\frac{1}{4}}l^{\frac{3}{4}}$$

Example: Implicit Function Theorem

Marginal rate of techincal substitution between k and l for

$$f(k,l) := 20k^{\frac{1}{4}}l^{\frac{3}{4}}$$

Fundamental result in Microeconomics that

$$MRTS_{k,l} := -\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial k}} = -\frac{3k}{l}$$

Example: Implicit Function Theorem

Marginal rate of techincal substitution between k and l for

$$f(k,l) := 20k^{\frac{1}{4}}l^{\frac{3}{4}}$$

Fundamental result in Microeconomics that

$$MRTS_{k,l} := -\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial k}} = -\frac{3k}{l}$$

This is an application of the *Implicit Function Theorem*

Example: Implicit Function Theorem

Marginal rate of techincal substitution between k and l for

$$f(k,l) := 20k^{\frac{1}{4}}l^{\frac{3}{4}}$$

Fundamental result in Microeconomics that

$$MRTS_{k,l} := -\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial k}} = -\frac{3k}{l}$$

This is an application of the *Implicit Function Theorem*

MRTS measures the slope $\frac{dk}{dl}$ along an isoquant curve of the production function where $f(k,l) = c$

Gradient Vector is perpendicular to the Level Set

Let f be a C^1 function on a neighbourhood of (y_0, x_0). Suppose (y_0, x_0) is a regular point of f. Then the gradient vector $\nabla f(y_0, x_0)$ is perpendicular to the level set of f at (y_0, x_0) .

Gradient Vector is perpendicular to the Level Set

Let f be a C^1 function on a neighbourhood of (y_0, x_0). Suppose (y_0, x_0) is a regular point of f. Then the gradient vector $\nabla f(y_0, x_0)$ is perpendicular to the level set of f at (y_0, x_0) .

See [SB Subsection 15.2]

System of Equations and Implicit Functions

Often we face a system of (generic) equations:

$$\begin{cases} f_1(y_1, \ldots, y_m; x_1, \ldots, x_n) = c_1 \\ \vdots \\ f_m(y_1, \ldots, y_m; x_1, \ldots, x_n) = c_m \end{cases}$$

or, in vector notation, $\boldsymbol{f}(\boldsymbol{y}; \boldsymbol{x}) = \boldsymbol{c}$.

System of Equations and Implicit Functions

Often we face a system of (generic) equations:

$$\begin{cases} f_1(y_1, \dots, y_m; x_1, \dots, x_n) = c_1 \\ \vdots \\ f_m(y_1, \dots, y_m; x_1, \dots, x_n) = c_m \end{cases}$$

or, in vector notation, $\boldsymbol{f}(\boldsymbol{y}; \boldsymbol{x}) = \boldsymbol{c}$.

$\boldsymbol{y}$ - 'endogenous variables' (often the primary variables of interest)

System of Equations and Implicit Functions

Often we face a system of (generic) equations:

$$\begin{cases} f_1(y_1, \ldots, y_m; x_1, \ldots, x_n) = c_1 \\ \quad\vdots \\ f_m(y_1, \ldots, y_m; x_1, \ldots, x_n) = c_m \end{cases}$$

or, in vector notation, $\boldsymbol{f}(\boldsymbol{y}; \boldsymbol{x}) = \boldsymbol{c}$.

$\boldsymbol{y}$ - 'endogenous variables' (often the primary variables of interest)

$\boldsymbol{x}$ - 'exogenous variables'

System of Equations and Implicit Functions

Often we face a system of (generic) equations:

$$\begin{cases} f_1(y_1, \ldots, y_m; x_1, \ldots, x_n) = c_1 \\ \vdots \\ f_m(y_1, \ldots, y_m; x_1, \ldots, x_n) = c_m \end{cases}$$ or, in vector notation, $\boldsymbol{f}(\boldsymbol{y}; \boldsymbol{x}) = \boldsymbol{c}$.

$\boldsymbol{y}$ - 'endogenous variables' (often the primary variables of interest)

$\boldsymbol{x}$ - 'exogenous variables'

Note that functionally this distinction is arbitrary- often distinguish between variables for a modelling/economic theoretical purpose

Example: Simple Linear System

Already seen an example of a general system

Example: Simple Linear System

Already seen an example of a general system

Given an $m \times n$ matrix A define $\boldsymbol{y} = A\boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbb{R}^n$

Example: Simple Linear System

Already seen an example of a general system

Given an $m \times n$ matrix A define $\boldsymbol{y} = A\boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbb{R}^n$

$f(\mathbf{x}, \mathbf{y}) = \mathbf{y} - A\boldsymbol{x}$ and $\boldsymbol{c} = 0$

Example: Simple Linear System

Already seen an example of a general system

Given an $m \times n$ matrix A define $\boldsymbol{y} = A\boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbb{R}^n$

$f(\mathbf{x}, \mathbf{y}) = \mathbf{y} - A\boldsymbol{x}$ and $\boldsymbol{c} = 0$

Each 'endogenous' variable a function only of 'exogenous' $x_1, ..., x_n$

Example: Simple Linear System

Already seen an example of a general system

Given an $m \times n$ matrix A define $\boldsymbol{y} = A\boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbb{R}^n$

$f(\mathbf{x}, \mathbf{y}) = \mathbf{y} - A\boldsymbol{x}$ and $\boldsymbol{c} = 0$

Each 'endogenous' variable a function only of 'exogenous' $x_1, ..., x_n$

Hence $\partial y_i / \partial x_j = a_{ij}$

Systems of Equations and Implicit Functions

Closed form solution for $\boldsymbol{y}$ often infeasible in non-linear models

Systems of Equations and Implicit Functions

Closed form solution for y often infeasible in non-linear models

Buy may imply **implicit** functions $y = y(x)$

Systems of Equations and Implicit Functions

Closed form solution for $\boldsymbol{y}$ often infeasible in non-linear models

Buy may imply **implicit** functions $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$

This often sufficient- when interest is how a change in parameters/exogenous variables x impacts endogenous variables y at a particular point

Jacobian Marices

Matrix of gradients of every f_i - **Jacobian matrix:**

$$J_{\boldsymbol{y}}^{\boldsymbol{f}} := \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_m} \end{pmatrix}, \; J_{\boldsymbol{x}}^{\boldsymbol{f}} := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Jacobian Marices

Matrix of gradients of every f_i - **Jacobian matrix:**

$$J_{\boldsymbol{y}}^{\boldsymbol{f}} := \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_m} \end{pmatrix}, \quad J_{\boldsymbol{x}}^{\boldsymbol{f}} := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Note these are functions, each element of the matrix is a particular partial derivative of some equation in the system with respect to some variable

Implicit Function Theorem (general)

Consider the system $\boldsymbol{f}(\boldsymbol{y};\boldsymbol{x}) = \boldsymbol{c}$ as described before. If the Jacobian matrix with respect to $\boldsymbol{y}$, $J_{\boldsymbol{y}}^{\boldsymbol{f}}$, is **invertible**, then

$$J_{\boldsymbol{x}}^{\boldsymbol{y}} := \begin{pmatrix} \dfrac{\partial y_1}{\partial x_1} & \cdots & \dfrac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial y_m}{\partial x_1} & \cdots & \dfrac{\partial y_m}{\partial x_n} \end{pmatrix} = -[J_{\boldsymbol{y}}^{\boldsymbol{f}}]^{-1} J_{\boldsymbol{x}}^{\boldsymbol{f}}.$$

Implicit Function Theorem (general)

Consider the system $\boldsymbol{f}(\boldsymbol{y};\boldsymbol{x}) = \boldsymbol{c}$ as described before. If the Jacobian matrix with respect to $\boldsymbol{y}$, $J^{\boldsymbol{f}}_{\boldsymbol{y}}$, is **invertible**, then

$$J^{\boldsymbol{y}}_{\boldsymbol{x}} := \begin{pmatrix} \dfrac{\partial y_1}{\partial x_1} & \cdots & \dfrac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial y_m}{\partial x_1} & \cdots & \dfrac{\partial y_m}{\partial x_n} \end{pmatrix} = -[J^{\boldsymbol{f}}_{\boldsymbol{y}}]^{-1} J^{\boldsymbol{f}}_{\boldsymbol{x}}.$$

Can show intuition using *total differentation* of the system

Implicit Function Theorem (general)

Consider the system $\boldsymbol{f}(\boldsymbol{y};\boldsymbol{x}) = \boldsymbol{c}$ as described before. If the Jacobian matrix with respect to $\boldsymbol{y}$, $J_{\boldsymbol{y}}^{\boldsymbol{f}}$, is **invertible**, then

$$J_{\boldsymbol{x}}^{\boldsymbol{y}} := \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix} = -[J_{\boldsymbol{y}}^{\boldsymbol{f}}]^{-1} J_{\boldsymbol{x}}^{\boldsymbol{f}}.$$

Can show intuition using *total differentation* of the system

See [SB Subsection 15.3]

Intution of Implicit Function Theorem of System

Derive $\left(\frac{dy_1}{dx}, \frac{dy_2}{dx}\right)$ from the following system: $(x > 0)$

$$\begin{aligned} x - 3y_1^2 + 2y_2 &= 0; \\ e^{y_2} - 2y_1 + \ln x &= 0. \end{aligned}$$

Intution of Implicit Function Theorem of System

Derive $\left(\frac{dy_1}{dx}, \frac{dy_2}{dx}\right)$ from the following system: $(x > 0)$

$$\begin{aligned} x - 3y_1^2 + 2y_2 &= 0; \\ e^{y_2} - 2y_1 + \ln x &= 0. \end{aligned}$$

Jacobian matrix $J_{\boldsymbol{y}}^{\boldsymbol{f}} = \begin{pmatrix} -6y_1 & 2 \\ -2 & e^{y_2} \end{pmatrix}$ needs to be invertible

$$\begin{vmatrix} -6y_1 & 2 \\ -2 & e^{y_2} \end{vmatrix} \neq 0 \Leftrightarrow 3y_1 e^{y_2} \neq 2$$

Intution of Implicit Function Theorem of System

Total differentiation of the system yields

$$
\begin{aligned}
dx + 6y_1 dy_1 + 2dy_2 &= 0 \\
e^{y_2} dy_2 - 2dy_1 + \frac{1}{x} dx &= 0.
\end{aligned}
$$

Intution of Implicit Function Theorem of System

Total differentiation of the system yields

$$\begin{aligned} dx + 6y_1 dy_1 + 2dy_2 &= 0 \\ e^{y_2} dy_2 - 2dy_1 + \frac{1}{x} dx &= 0. \end{aligned}$$

'Solving' this system of dy_1, dy_2 and dx yields:

$$\frac{dy_1}{dx} = \frac{xe^{y_2} - 2}{2y_1(3e^{y_2} - 2)}$$

$$\frac{dy_2}{dx} = \frac{x - 3y_1}{x(3y_1 e^{y_2} - 2)}$$

Second Order Partial Derivatives

Each partial derivative $\dfrac{\partial f(\boldsymbol{y})}{\partial y_i}$ is also a multivariate function on $\mathbb{R}^n$ with its own derivative, i.e. the 2^{nd} **order partial derivatives** of f

$$\frac{\partial^2 f(\boldsymbol{y})}{\partial y_j \partial y_i} := \frac{\partial}{\partial y_j}\left(\frac{\partial f(\boldsymbol{y})}{\partial y_i}\right)$$

Second Order Partial Derivatives

Each partial derivative $\dfrac{\partial f(\boldsymbol{y})}{\partial y_i}$ is also a multivariate function on $\mathbb{R}^n$ with its own derivative, i.e. the 2^{nd} **order partial derivatives** of f

$$\frac{\partial^2 f(\boldsymbol{y})}{\partial y_j \partial y_i} := \frac{\partial}{\partial y_j}\left(\frac{\partial f(\boldsymbol{y})}{\partial y_i}\right)$$

Sometimes denote $\dfrac{\partial^2 f(\boldsymbol{y})}{\partial y_j \partial y_i}$ as $f_{ji}(\boldsymbol{y})$ for brevity

Second Order Partial Derivatives

Each partial derivative $\frac{\partial f(\boldsymbol{y})}{\partial y_i}$ is also a multivariate function on $\mathbb{R}^n$ with its own derivative, i.e. the 2^{nd} **order partial derivatives** of f

$$\frac{\partial^2 f(\boldsymbol{y})}{\partial y_j \partial y_i} := \frac{\partial}{\partial y_j}\left(\frac{\partial f(\boldsymbol{y})}{\partial y_i}\right)$$

Sometimes denote $\frac{\partial^2 f(\boldsymbol{y})}{\partial y_j \partial y_i}$ as $f_{ji}(\boldsymbol{y})$ for brevity

A function f is defined as C^1 if all its partial derivatives are continuous throughout its domain, and is C^2 if all its 2^{nd} order partial derivatives are continuous as well

Hessian Matrix

Hessian Matrices and Young's Theorem

Hessian matrix is defined as the matrix of 2^{nd} order partial derivatives of a multivariate function f on $\mathbb{R}^n$:

$$H^f(\boldsymbol{y}) := \left(\frac{\partial^2 f(\boldsymbol{y})}{\partial y_i \partial y_j}\right)_{n\times n} \quad \text{or} \quad (f_{ij}(\boldsymbol{y}))_{n\times n}$$

Hessian Matrices and Young's Theorem

Hessian matrix is defined as the matrix of 2^{nd} order partial derivatives of a multivariate function f on $\mathbb{R}^n$:

$$H^f(\boldsymbol{y}) := \left(\frac{\partial^2 f(\boldsymbol{y})}{\partial y_i \partial y_j}\right)_{n\times n} \quad \text{or} \quad (f_{ij}(\boldsymbol{y}))_{n\times n}$$

Young's Theorem: If $f : S \subset \mathbb{R}^n \mapsto \mathbb{R}$ is a C^2 function, then at every point $\boldsymbol{y} \in S$,

$\dfrac{\partial^2 f(\boldsymbol{y})}{\partial y_i \partial y_j} = \dfrac{\partial^2 f(\boldsymbol{y})}{\partial y_j \partial y_i}$ for every i, j, i.e. $H^f(\boldsymbol{y})$ is a **symmetric** matrix.

Hessian Matrices and Young's Theorem

Hessian matrix is defined as the matrix of 2^{nd} order partial derivatives of a multivariate function f on $\mathbb{R}^n$:

$$H^f(\boldsymbol{y}) := \left(\frac{\partial^2 f(\boldsymbol{y})}{\partial y_i \partial y_j}\right)_{n\times n} \text{ or } (f_{ij}(\boldsymbol{y}))_{n\times n}$$

Young's Theorem: If $f : S \subset \mathbb{R}^n \mapsto \mathbb{R}$ is a C^2 function, then at every point $\boldsymbol{y} \in S$,

$\dfrac{\partial^2 f(\boldsymbol{y})}{\partial y_i \partial y_j} = \dfrac{\partial^2 f(\boldsymbol{y})}{\partial y_j \partial y_i}$ for every i, j, i.e. $H^f(\boldsymbol{y})$ is a **symmetric** matrix.

See [SB Subsection 14.8]

Example: Deriving Hessian Matrix

$$f(y_1, y_2, y_3) = 100 - 2y_1^2 - y_2^2 - 3y_3 - y_1 y_2 - e^{y_1+y_2+y_3}:$$

$$H^f(y_1, y_2, y_3) := \begin{pmatrix} \frac{\partial^2 f(\boldsymbol{y})}{\partial y_1^2} & \frac{\partial^2 f(\boldsymbol{y})}{\partial y_1 \partial y_2} & \frac{\partial^2 f(\boldsymbol{y})}{\partial y_1 \partial y_3} \\ \frac{\partial^2 f(\boldsymbol{y})}{\partial y_2 \partial y_1} & \frac{\partial^2 f(\boldsymbol{y})}{\partial y_2^2} & \frac{\partial^2 f(\boldsymbol{y})}{\partial y_2 \partial y_3} \\ \frac{\partial^2 f(\boldsymbol{y})}{\partial y_3 \partial y_1} & \frac{\partial^2 f(\boldsymbol{y})}{\partial y_3 \partial y_2} & \frac{\partial^2 f(\boldsymbol{y})}{\partial y_3^2} \end{pmatrix}$$

Example: Deriving Hessian Matrix

$$f(y_1, y_2, y_3) = 100 - 2y_1^2 - y_2^2 - 3y_3 - y_1y_2 - e^{y_1+y_2+y_3}:$$

$$\begin{aligned}
& H^f(y_1, y_2, y_3) \\
:= & \begin{pmatrix} \frac{\partial^2 f(\boldsymbol{y})}{\partial y_1^2} & \frac{\partial^2 f(\boldsymbol{y})}{\partial y_1 \partial y_2} & \frac{\partial^2 f(\boldsymbol{y})}{\partial y_1 \partial y_3} \\ \frac{\partial^2 f(\boldsymbol{y})}{\partial y_2 \partial y_1} & \frac{\partial^2 f(\boldsymbol{y})}{\partial y_2^2} & \frac{\partial^2 f(\boldsymbol{y})}{\partial y_2 \partial y_3} \\ \frac{\partial^2 f(\boldsymbol{y})}{\partial y_3 \partial y_1} & \frac{\partial^2 f(\boldsymbol{y})}{\partial y_3 \partial y_2} & \frac{\partial^2 f(\boldsymbol{y})}{\partial y_3^2} \end{pmatrix} \\
= & \begin{pmatrix} -4 - e^{y_1+y_2+y_3} & -1 - e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} \\ -1 - e^{y_1+y_2+y_3} & -2 - e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} \\ -e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} \end{pmatrix}
\end{aligned}$$

Concave/Convex Functions

Concave Functions

A function $f : S \subset \mathbb{R}^n \mapsto \mathbb{R}$ is **concave** on S if

$f(\lambda \boldsymbol{y}+(1-\lambda)\boldsymbol{x}) \geq \lambda f(\boldsymbol{y})+(1-\lambda)f(\boldsymbol{x})$ for all $\boldsymbol{y}, \boldsymbol{x} \in S$ and $\lambda \in [0, 1]$.

Concave Functions

A function $f : S \subset \mathbb{R}^n \mapsto \mathbb{R}$ is **concave** on S if

$f(\lambda \boldsymbol{y} + (1-\lambda)\boldsymbol{x}) \geq \lambda f(\boldsymbol{y}) + (1-\lambda) f(\boldsymbol{x})$ for all $\boldsymbol{y}, \boldsymbol{x} \in S$ and $\lambda \in [0, 1]$.

Concave/convex functions are defined on **convex sets**

Concave Functions

A function $f : S \subset \mathbb{R}^n \mapsto \mathbb{R}$ is **concave** on S if

$f(\lambda \boldsymbol{y}+(1-\lambda)\boldsymbol{x}) \geq \lambda f(\boldsymbol{y})+(1-\lambda)f(\boldsymbol{x})$ for all $\boldsymbol{y}, \boldsymbol{x} \in S$ and $\lambda \in [0, 1]$.

Concave/convex functions are defined on **convex sets**

If y_1 and y_2 in set S then so is $\lambda y_1 + (1 - \lambda)y_2$ for all $\lambda \in [0, 1]$

Convex Functions

A function f is **convex** on S if

$f(\lambda \boldsymbol{y} + (1-\lambda)\boldsymbol{x}) \leq \lambda f(\boldsymbol{y}) + (1-\lambda) f(\boldsymbol{x})$ for all $\boldsymbol{y}, \boldsymbol{x} \in S$ and $\lambda \in [0, 1]$.

Hessian Matrices and Concavity/Convexity

A C^2 function $f : S \subset \mathbb{R}^n \mapsto \mathbb{R}$ is **concave** if and only if

$$H^f(\boldsymbol{y}) \text{ is } \textbf{negative-semidefinite} \text{ for all } \boldsymbol{y} \in S.$$

Hessian Matrices and Concavity/Convexity

A C^2 function $f : S \subset \mathbb{R}^n \mapsto \mathbb{R}$ is **concave** if and only if

$$H^f(\boldsymbol{y}) \text{ is } \textbf{negative-semidefinite} \text{ for all } \boldsymbol{y} \in S.$$

f is **convex** if and only if

$$H^f(\boldsymbol{y}) \text{ is } \textbf{positive-semidefinite} \text{ for all } \boldsymbol{y} \in S.$$

Hessian Matrices and Concavity/Convexity

A C^2 function $f : S \subset \mathbb{R}^n \mapsto \mathbb{R}$ is **concave** if and only if

$$H^f(\boldsymbol{y}) \text{ is } \textbf{negative-semidefinite} \text{ for all } \boldsymbol{y} \in S.$$

f is **convex** if and only if

$$H^f(\boldsymbol{y}) \text{ is } \textbf{positive-semidefinite} \text{ for all } \boldsymbol{y} \in S.$$

For univariate functions, conditions reduced to *non-positive* (*non-negative*) $f''(y)$ for concavity (convexity)

- *e.g.*, $-2x^2$ is concave and $y^2 + y + 2$ is convex

Example of Checking Concavity/Convexity

$$f(y_1, y_2, y_3) = 100 - 2y_1^2 - y_2^2 - 3y_3 - y_1 y_2 - e^{y_1+y_2+y_3}$$

$$H^f(y_1, y_2, y_3) = \begin{pmatrix} -4 - e^{y_1+y_2+y_3} & -1 - e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} \\ -1 - e^{y_1+y_2+y_3} & -2 - e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} \\ -e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} \end{pmatrix}$$

Example of Checking Concavity/Convexity

$$f(y_1, y_2, y_3) = 100 - 2y_1^2 - y_2^2 - 3y_3 - y_1y_2 - e^{y_1+y_2+y_3}$$

$$H^f(y_1, y_2, y_3) = \begin{pmatrix} -4 - e^{y_1+y_2+y_3} & -1 - e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} \\ -1 - e^{y_1+y_2+y_3} & -2 - e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} \\ -e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} \end{pmatrix}$$

Compute the leading principal minors:

$$\begin{aligned} D_1^* &= -4 - e^{y_1+y_2+y_3} < 0, \\ D_2^* &= 7 + 4e^{y_1+y_2+y_3} > 0, \\ D_3^* &= -7e^{y_1+y_2+y_3} < 0 \end{aligned}$$

Hence H^f is *negative definite* and f is *concave*

Example of Checking Concavity/Convexity

$$f(y_1, y_2, y_3) = 100 - 2y_1^2 - y_2^2 - 3y_3 - y_1y_2 - e^{y_1+y_2+y_3}$$

$$H^f(y_1, y_2, y_3) = \begin{pmatrix} -4 - e^{y_1+y_2+y_3} & -1 - e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} \\ -1 - e^{y_1+y_2+y_3} & -2 - e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} \\ -e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} & -e^{y_1+y_2+y_3} \end{pmatrix}$$

Compute the leading principal minors:

$$\begin{aligned} D_1^* &= -4 - e^{y_1+y_2+y_3} < 0, \\ D_2^* &= 7 + 4e^{y_1+y_2+y_3} > 0, \\ D_3^* &= -7e^{y_1+y_2+y_3} < 0 \end{aligned}$$

Hence H^f is *negative definite* and f is *concave*

See [SB Subsection 21.1 and SHSS Sections 2.2. and 2.3]

Mean Value Theorem

Mean Value Theorem

If f is C^1 and for any $\boldsymbol{y}, \boldsymbol{x}$ if the domain contains $\lambda \boldsymbol{y}$ +(1-λ)$\boldsymbol{x}$, for all $\lambda \in (0, 1)$ then for some $\boldsymbol{w}$ in this set

$$f(\boldsymbol{y}) - f(\boldsymbol{x}) = \nabla f(\boldsymbol{w}) \cdot (\boldsymbol{y} - \boldsymbol{x})$$

Mean Value Theorem

If f is C^1 and for any $\boldsymbol{y}, \boldsymbol{x}$ if the domain contains $\lambda \boldsymbol{y}$ +(1-λ)$\boldsymbol{x}$, for all $\lambda \in (0,1)$ then for some $\boldsymbol{w}$ in this set

$$f(\boldsymbol{y}) - f(\boldsymbol{x}) = \nabla f(\boldsymbol{w}) \cdot (\boldsymbol{y} - \boldsymbol{x})$$

See [SB pp. 824-826 and SHSS p. 48]

Taylor Series

k'th Order Taylor Expansion (univariate)

If $f : S \in \mathbb{R} \mapsto \mathbb{R}$ is C^{k+1} and its domain contains the interval between y and a,

$$\begin{aligned} f(y) &= f(a) + \frac{f'(a)}{1!}(y-a) + \frac{f''(a)}{2!}(y-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(y-a)^k \\ &\quad + \frac{f^{(k+1)}(c)}{(k+1)!}(y-a)^{k+1}, \text{ for some } c \text{ between } y \text{ and } a. \end{aligned}$$

k'th Order Taylor Expansion (univariate)

If $f : S \in \mathbb{R} \mapsto \mathbb{R}$ is C^{k+1} and its domain contains the interval between y and a,

$$\begin{aligned} f(y) &= f(a) + \frac{f'(a)}{1!}(y-a) + \frac{f''(a)}{2!}(y-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(y-a)^k \\ &+ \frac{f^{(k+1)}(c)}{(k+1)!}(y-a)^{k+1}, \text{ for some } c \text{ between } y \text{ and } a. \end{aligned}$$

When a is close to y, we can approximate $f(y)$ with the polynomial on the right hand side by substituting c with a

Second Order Taylor Expansion (multivariate)

If f is a C^2 function on $\mathbb{R}^n$ and its domain contains all the convex combinations of two points $\boldsymbol{y}$ and $\boldsymbol{a}$,

$$\begin{aligned} f(\boldsymbol{y}) &= f(\boldsymbol{a}) + \sum_{i=1}^{n} \frac{\partial f(\boldsymbol{a})}{\partial y_i}(\boldsymbol{y}_i - \boldsymbol{a}_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f(\boldsymbol{c})}{\partial y_i \partial y_j}(\boldsymbol{y}_i - \boldsymbol{a}_i)(\boldsymbol{y}_j - \boldsymbol{a}_j) \\ &= f(\boldsymbol{a}) + \nabla f(\boldsymbol{a}) \cdot (\boldsymbol{y} - \boldsymbol{a}) + \frac{1}{2}(\boldsymbol{y} - \boldsymbol{a})^{\mathsf{T}} H^f(\boldsymbol{c})(\boldsymbol{y} - \boldsymbol{a}), \end{aligned}$$

for some $\boldsymbol{c}$ that is a convex combination of $\boldsymbol{y}$ and $\boldsymbol{a}$.

Second Order Taylor Expansion (multivariate)

If f is a C^2 function on $\mathbb{R}^n$ and its domain contains all the convex combinations of two points $\boldsymbol{y}$ and $\boldsymbol{a}$,

$$\begin{aligned} f(\boldsymbol{y}) &= f(\boldsymbol{a}) + \sum_{i=1}^{n} \frac{\partial f(\boldsymbol{a})}{\partial y_i}(\boldsymbol{y}_i - \boldsymbol{a}_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f(\boldsymbol{c})}{\partial y_i \partial y_j}(\boldsymbol{y}_i - \boldsymbol{a}_i)(\boldsymbol{y}_j - \boldsymbol{a}_j) \\ &= f(\boldsymbol{a}) + \nabla f(\boldsymbol{a}) \cdot (\boldsymbol{y} - \boldsymbol{a}) + \frac{1}{2}(\boldsymbol{y} - \boldsymbol{a})^\intercal H^f(\boldsymbol{c})(\boldsymbol{y} - \boldsymbol{a}), \end{aligned}$$

for some $\boldsymbol{c}$ that is a convex combination of $\boldsymbol{y}$ and $\boldsymbol{a}$.

Namely $\boldsymbol{c}$ is on the line segment connnecting $\boldsymbol{y}$ and $\boldsymbol{a}$

See [SB Subsection 30.1 and 30.2 and SHSS Section 2.6]

Linearization of Systems of Functions

$$f(\boldsymbol{y}) \approx f(\boldsymbol{a}) + \nabla f(\boldsymbol{a}) \cdot (\boldsymbol{y} - \boldsymbol{a}) \quad \text{linear approximation of f(y)}$$

Linearization of Systems of Functions

$$f(\boldsymbol{y}) \approx f(\boldsymbol{a}) + \nabla f(\boldsymbol{a}) \cdot (\boldsymbol{y} - \boldsymbol{a}) \quad \text{linear approximation of f(y)}$$

Can apply to each function in a generic system

$$\begin{cases} f_1(y_1, \ldots, y_n) = c_1 \\ \qquad \vdots \\ f_n(y_1, \ldots, y_n) = c_n \end{cases} \quad \text{or simply } \boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{c}.$$

Linearization of Systems of Functions

$$f(\boldsymbol{y}) \approx f(\boldsymbol{a}) + \nabla f(\boldsymbol{a}) \cdot (\boldsymbol{y} - \boldsymbol{a}) \quad \text{linear approximation of f(y)}$$

Can apply to each function in a generic system

$$\begin{cases} f_1(y_1, \ldots, y_n) = c_1 \\ \quad\vdots \\ f_n(y_1, \ldots, y_n) = c_n \end{cases} \quad \text{or simply } \boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{c}.$$

(locally) linearize the system around an point $\boldsymbol{y}^*$:

$$\begin{cases} f_1(\boldsymbol{y}^*) + \nabla f_1(\boldsymbol{y}^*) \cdot (\boldsymbol{y} - \boldsymbol{y}^*) \approx c_1 \\ \quad\cdots\cdots\cdots\cdots \\ f_n(\boldsymbol{y}^*) + \nabla f_n(\boldsymbol{y}^*) \cdot (\boldsymbol{y} - \boldsymbol{y}^*) \approx c_n \end{cases} \quad \text{or } J^{\boldsymbol{f}}(\boldsymbol{y}^*)(\boldsymbol{y} - \boldsymbol{y}^*) \approx \boldsymbol{c} - \boldsymbol{f}(\boldsymbol{y}^*).$$

Integration

Often need to compute *expectations* of a random function

Often need to compute *expectations* of a random function

Simplest example is the mean of a variable ('the expectation of X')

Often need to compute *expectations* of a random function

Simplest example is the mean of a variable ('the expectation of X')

Expectation Operator: For continuous function g and random variable X with pdf $f(x)$ then $E[g(X)] := \int_{-\infty}^{\infty} g(x)f(x)dx$

Integration

Often need to compute *expectations* of a random function

Simplest example is the mean of a variable ('the expectation of X')

Expectation Operator: For continuous function g and random variable X with pdf $f(x)$ then $E[g(X)] := \int_{-\infty}^{\infty} g(x)f(x)dx$

Often used to form future expected utility/outputs for some saving/investment path (important for optimal control problem)

Indefinite and Definite Integrals

An **indefinite integral** of a (univariate) function $f(y)$ is given by

$$\int f(y)dy = F(y) + C, \text{ where } F'(y) = f(y)$$

F is also called the **primitive function (or inverse derivative)** of f

Indefinite and Definite Integrals

An **indefinite integral** of a (univariate) function $f(y)$ is given by

$$\int f(y)dy = F(y) + C, \text{ where } F'(y) = f(y)$$

F is also called the **primitive function (or inverse derivative)** of f

The **definite integral** of a (univariate) function $f(y)$ over an interval $[a, b]$ is given by

$$\int_a^b f(y)dy = F(b) - F(a), \text{ where } F'(y) = f(y) \text{ for all } y \in (a, b)$$

This is also known as the **Fundamental Theorem of Calculus**

Integration by Parts

One useful rule for integration is **integration by parts**:

$$\int f(y)g(y)dy = f(y)G(y) - \int f'(y)G(y)dy + C, \text{ with } G'(y) = g(y)$$

Integration by Parts

One useful rule for integration is **integration by parts**:

$$\int f(y)g(y)dy = f(y)G(y) - \int f'(y)G(y)dy + C, \text{ with } G'(y) = g(y)$$

It comes from the *product rule* for differentiation:

$$\frac{d}{dy}[f(y)G(y)] = f'(y)G(y) + f(y)g(y)$$

See [SB pp. 887-890]

Example: Integration by Parts

$$\int e^{2x}(3x-1)dx \;=\; (3x-1)\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}3dx + C$$

Example: Integration by Parts

$$\int e^{2x}(3x-1)dx = (3x-1)\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}3dx + C$$
$$= (3x-1)\frac{e^{2x}}{2} - \frac{3}{4}e^{2x} + C$$

Example: Integration by Parts

$$\begin{aligned}
\int e^{2x}(3x-1)dx &= (3x-1)\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} 3dx + C \\
&= (3x-1)\frac{e^{2x}}{2} - \frac{3}{4}e^{2x} + C \\
&= \frac{e^{2x}}{4}(6x-5) + C.
\end{aligned}$$

Integration by Substitution

Another useful method for integration is **integration by substitution** or **change of variable**:

$$\int f(g(y))g'(y)dy = \int f(u)du, \text{ where } u = g(y)$$

Integration by Substitution

Another useful method for integration is **integration by substitution** or **change of variable**:

$$\int f(g(y))g'(y)dy = \int f(u)du, \text{ where } u = g(y)$$

For a *definite integral*, the bounds need to be adjusted accordingly:

$$\int_a^b f(g(y))g'(y)dy = \int_{g(a)}^{g(b)} f(u)du$$

Example: Integration by Susbtitution

$$\int_0^1 \frac{4y^2}{\sqrt{4-y^2}}dy \;=\; \int_2^{\sqrt{3}} 4(u^2-4)du, \text{ where } u = \sqrt{4-y^2}$$

Example: Integration by Susbtitution

$$\int_0^1 \frac{4y^2}{\sqrt{4-y^2}}dy = \int_2^{\sqrt{3}} 4(u^2-4)du, \text{ where } u = \sqrt{4-y^2}$$

$$= \left(\frac{4u^3}{3} - 16u\right)\Bigg|_{u=2}^{u=\sqrt{3}} =$$

Example: Integration by Susbtitution

$$\int_0^1 \frac{4y^2}{\sqrt{4-y^2}}dy = \int_2^{\sqrt{3}} 4(u^2-4)du, \text{ where } u = \sqrt{4-y^2}$$

$$= \left(\frac{4u^3}{3} - 16u\right)\Big|_{u=2}^{u=\sqrt{3}} = \frac{64}{3} - 12\sqrt{3}.$$

Leibniz's Rule

Often encounter paramaters that may change (e.g. over time)

Leibniz's Rule

Often encounter paramaters that may change (e.g. over time)

How does the value of an integral change if the parameters change?

Leibniz's Rule

Often encounter paramaters that may change (e.g. over time)

How does the value of an integral change if the parameters change?

Leibniz's Rule If f, a and b are all C^1 functions, then

$$\frac{d}{ds}\left[\int_{a(s)}^{b(s)} f(x,s)dx\right] = f(b(s),s)b'(x) - f(a(s),s)a'(s) + \int_{a(s)}^{b(s)} \frac{\partial f(x,s)}{\partial s}dx.$$

Leibniz's Rule

Can derive using *chain rule* and the *fundamental theorem of calculus*

Let $H(a, b, s) := \int_a^b f(x, s)dx$, then

$\int_{a(s)}^{b(s)} f(x, s)dx := H(a(s), b(s), s)$

$$\frac{d}{ds}[H(a(s), b(s), s)] = \frac{\partial H}{\partial a}a'(s) + \frac{\partial H}{\partial b}b'(s) + \frac{\partial H}{\partial x}, \text{ with}$$

Leibniz's Rule

Can derive using *chain rule* and the *fundamental theorem of calculus*

Let $H(a, b, s) := \int_a^b f(x, s)dx$, then

$\int_{a(s)}^{b(s)} f(x, s)dx := H(a(s), b(s), s)$

$$\frac{d}{ds}[H(a(s), b(s), s)] = \frac{\partial H}{\partial a}a'(s) + \frac{\partial H}{\partial b}b'(s) + \frac{\partial H}{\partial x}, \text{ with}$$
$$\frac{\partial H}{\partial a} = -f(a, s), \ \frac{\partial H}{\partial b} = f(b, s), \ \frac{\partial H}{\partial s} := \int_a^b \frac{\partial f(x, s)}{\partial x}ds.$$

Example: Leibniz's Rule

Derive the derivative of $F(s) := \int_0^1 \frac{1}{1+sx} dx$:

Example: Leibniz's Rule

Derive the derivative of $F(s) := \int_0^1 \frac{1}{1+sx}dx$:

$$F'(s) \quad = \quad \int_0^1 -\frac{x}{(1+sx)^2}dx$$

Example: Leibniz's Rule

Derive the derivative of $F(s) := \int_0^1 \frac{1}{1+sx} dx$:

$$
\begin{aligned}
F'(s) &= \int_0^1 -\frac{x}{(1+sx)^2} dx \\
&= \left.\frac{x}{s(1+sx)}\right|_{x=0}^{x=1} - \int_0^1 \frac{1}{s(1+sx)} dx \textit{ [integration by parts]}
\end{aligned}
$$

Example: Leibniz's Rule

Derive the derivative of $F(s) := \int_0^1 \frac{1}{1+sx} dx$:

$$
\begin{aligned}
F'(s) &= \int_0^1 -\frac{x}{(1+sx)^2} dx \\
&= \left.\frac{x}{s(1+sx)}\right|_{x=0}^{x=1} - \int_0^1 \frac{1}{s(1+sx)} dx \text{ [integration by parts]} \\
&= \frac{1}{s(1+s)} - \frac{\ln(1+s)}{s^2}.
\end{aligned}
$$

Example: Leibniz's Rule

Derive the derivative of $F(s) := \int_0^1 \frac{1}{1+sx}dx$:

$$\begin{aligned} F'(s) &= \int_0^1 -\frac{x}{(1+sx)^2}dx \\ &= \frac{x}{s(1+sx)}\Big|_{x=0}^{x=1} - \int_0^1 \frac{1}{s(1+sx)}dx \textit{ [integration by parts]} \\ &= \frac{1}{s(1+s)} - \frac{\ln(1+s)}{s^2}. \end{aligned}$$

Compute the derivate of $G(s) = \int_s^{s^2} \frac{sx^2}{2}dx$:

$$G'(s) = \frac{s^5}{2}\cdot 2s - \frac{s^3}{2}\cdot 1 + \int_s^{s^2} \frac{s^2}{2}dx$$

Example: Leibniz's Rule

Derive the derivative of $F(s) := \int_0^1 \frac{1}{1+sx} dx$:

$$\begin{aligned} F'(s) &= \int_0^1 -\frac{x}{(1+sx)^2} dx \\ &= \frac{x}{s(1+sx)}\Big|_{x=0}^{x=1} - \int_0^1 \frac{1}{s(1+sx)} dx \text{ [integration by parts]} \\ &= \frac{1}{s(1+s)} - \frac{\ln(1+s)}{s^2}. \end{aligned}$$

Compute the derivate of $G(s) = \int_s^{s^2} \frac{sx^2}{2} dx$:

$$\begin{aligned} G'(s) &= \frac{s^5}{2} \cdot 2s - \frac{s^3}{2} \cdot 1 + \int_s^{s^2} \frac{s^2}{2} dx \\ &= s^6 - \frac{x^3}{2} + \frac{s^6 - x^3}{6} = \frac{7s^6}{6} - \frac{2s^3}{3}. \end{aligned}$$