Mathematical Economics|EC3304

Linear Algebra

Nicky Grant (Semester 2, 2024/2025)

Reading

Further Mathematics for Economic Analysis, Sydseater et al 2^{nd} Edition, 2008 [**SHSS**]

Mathematics for Economists, Simon and Blume 1^{st} Edition, [**SB**]

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Optional - *Essential Mathematics for Economic Analysis*, Sydseater et al., 6th Edition

Introductory level, compliments well SHSS if you need more explanation of core concepts

Overview of Topics

Preliminary Maths - Lectures 1-4

Linear Algebra || (Multivariate) Calculus || Elementary Topology

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Static Optimization- Lectures 5-6

Unconstrained Problems || Constrained Problems || Comparative Statics

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Dynamic Optimization-Lecture 7-8

(Ordinary) Differential Equations || Continuous-time Problems

Maths, Language & the Visual Sign

Mathematics and Language

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Employs its own signs to distinguish/denote various entities

Seemingly more rigid and rigorous than lexical language-ultimately any result has meaning only through interpretation/context

The Visual Sign

$$+$$

$$\sum_{i=1}^{n} x_i$$

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$$x_1 + x_2 + x_3 + \cdots + x_n$$

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$$x_1 + x_2 + x_3 + \cdots + x_n$$

Total sum of all elements of some variable x for each entity 1 to n

VISUAL

VISUAL
LEXICAL

VISUAL
LEXICAL
ANALYTICAL

Motivating Economic Examples

Example: IS-LM Model (SB pp.115-117)

C - Consumption, G - Government Expenditure, I - Investment, Y - National Income, r - Interest Rate, M_s - Money Supply, M_d - Money Demand.

$$
\begin{aligned}
C &= bY \\
Y &= C + I + G \\
I &= I^0 - ar \\
M_s &= mY + M^0 - hr \\
M_d &= M_s
\end{aligned}
$$

Utility Maximisation s.t Budget Constraints

Two goods (x, y) with prices (p_x, p_y) with income M

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Continuous time Consumption-Saving problem

Individual with initial wealth a_0 chooses a consumption path $\{c(t)\}_{t\geq 0}$ to maximize their lifetime utility:

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$$\text{s.t.} \quad da/dt = ra(t) - c(t) \text{ with } a(0) = a_0 > 0,$$

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The 'constraint' above is in the form of a (linear) *differential equation*

Euclidean Spaces

Vectors

n-vector: for any n-tuple of (real) numbers $x_1, \cdots x_n$

$$\mathbf{x} := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

Vector Addition/Subtraction

For two n-vectors $\mathbf{x}, \mathbf{y}$,

$$\mathbf{x} \pm \mathbf{y} := (x_1 \pm y_1, x_2 \pm y_2, \cdots, x_n \pm y_n)$$

Scalar Multiplication

For any n-vector $\mathbf{x}$ and for any scalar $k \in \mathbb{R}$

$$k\mathbf{x} := (kx_1, kx_2, \cdots, kx_n)$$

Inner Product

The **inner product** of two n-vectors **x**, **y**,

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EXAMPLE:

$$\mathbf{x} = (2, 3, 1), \qquad \mathbf{y} = (-4, 2, 1)$$

$$\Rightarrow \quad \mathbf{x} \cdot \mathbf{y} = 2 \times (-4) + 3 \times 2 + 1 \times 1 = -1$$

Vector Norm

The **Vector Norm/Length:** for any n-vector **x**

$$\|\mathbf{x}\| := \sqrt{\mathbf{x} \cdot \mathbf{x}} = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}}$$

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Cauchy-Schwarz inequality

$$|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$$

Example: Least Squares Minimisation

Loss function $L(b, \mathbf{x}, \mathbf{y}) := \sum_{i=1}^{n} (y_i - bx_i)^2$

where $b \in \mathbb{R}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$.

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Find b that minimizes the loss function $L(b, \mathbf{x}, \mathbf{y})$.

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Tutorial 1 Question 1. (Also solved in the textbook reading asssigned).

Angle Between Two Vectors (no proof)

Theorem: The angle (θ) between two n-vectors $\mathbf{x}, \mathbf{y}$ satisfies

$$\cos(\theta) = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|}$$

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Two n-vectors $\mathbf{x}, \mathbf{y}$ **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if

$$\mathbf{x} \cdot \mathbf{y} = 0$$

Hyperplanes and Normal Vector

Hyperplane passing through $\mathbf{a} \in \mathbb{R}^n$ and orthogonal to some nonzero vector $\mathbf{p} \in \mathbb{R}^n$ ($\mathbf{p}$ is called the **normal vector**)

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EXAMPLE: budget constraint in pure exchange economy

$$p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = p_1 w_1 + p_1 w_2 + \cdots + p_n w_n$$

$$\mathbf{p} \cdot (\mathbf{x} - \mathbf{w}) = 0$$

Matrices and Linear Systems

Matrices

Matrix A is an $m \times n$ rectangular array of (real) numbers

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

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$A^\intercal := (a_{ji})_{n \times m}$ is the **transpose** of A

Matrix Operations

Addition/Subtraction for any $A := (a_{ij})_{m \times n}, B := (b_{ij})_{m \times n}$

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Scalar Multiplication for any $A := (a_{ij})_{m\times n},\ k \in \mathbb{R}$

$$kA := (ka_{ij})_{m\times n}$$

Matrix Multiplication

Matrix Multiplication for any $A := (a_{ij})_{m \times n}, B := (b_{ij})_{n \times p}$

$$AB := (c_{ij})_{m \times p} \text{ with } c_{ij} := \sum_{r=1}^{n} a_{ir} b_{rj}$$

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EXAMPLE

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix} \qquad AB = \begin{pmatrix} 2 & 5 \\ 0 & -3 \end{pmatrix},$$

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Special Square Matrices

Identity matrix often denoted $\boldsymbol{I}$

$$\boldsymbol{I} := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n \times n}$$

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Diagonal matrix - A with $a_{ij} = 0$ for all $i \neq j$, often written as $diag(a_{11}, \ldots, a_{nn})$

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Idempotent matrix - $A^2 = A$

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Lower triangular matrix - A with $a_{ij} = 0$ for all $i < j$

Symmetric matrix - $A = A^{\intercal}$

Idempotent matrix - $A^2 = A$

Orthogonal matrix - $AA^{\intercal} = A^{\intercal}A = I$

Linear Independence

The vectors $\boldsymbol{a}^1, \boldsymbol{a}^2, \cdots, \boldsymbol{a}^n$ in $\mathbb{R}^m$ are **linearly dependent** if there exist real numbers $c_1, c_2, \cdots, c_n$, not all zero, such that

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$c_1\boldsymbol{a}^1 + c_2\boldsymbol{a}^2 + \cdots + c_n\boldsymbol{a}^n$ is called a *linear combination* of $\boldsymbol{a}^1, \boldsymbol{a}^2, \cdots, \boldsymbol{a}^n$

Rank of a Matrix

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The process is sometimes called **Gaussian elimination**

Example: Finding the Rank of a 3×3 Matrix

Find the rank of $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{pmatrix}$

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Find the rank of $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{pmatrix}$

Adding (-2) times the first row to the second row and (-1) times the first row to the third row

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{pmatrix}$$

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Adding the second row to the third row

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Exists one linearly dependent combination of columns. Hence the rank is 2

EXAMPLE: Determinant of 2×2 matrix

$$\left|(a_{ij})_{2\times 2}\right| = a_{11}a_{22} - a_{21}a_{12}$$

Determinant of a General Square Matrix

The **determinant** of a (square) matrix $A := (a_{ij})_{n\times n}$

$$|A| := a_{11} \text{ when } n = 1;$$

$$|A| := \sum_{j=1}^{n} a_{ij}A_{ij},\ \forall i = 1, \ldots, n, \text{ when } n > 1,$$

A_{ij} is determinant of the submatrix of A with i-th row & j-th column deleted (multiplied by $(-1)^{i+j}$).

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A_{ij} is the **co-factor** of a_{ij} in the expansion of $|A|$ above

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Choose (for instance) the first row of A to start the computation:

$$|A| = (-1)^{1+1} \cdot (-2) \cdot \begin{vmatrix} 2 & -1 \\ -4 & 4 \end{vmatrix} + (-1)^{1+2} \cdot 0 \cdot \begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix}$$
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\begin{aligned}
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& +(-1)^{1+3} \cdot 1 \cdot \begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix} \\
= \quad & (-2) \begin{vmatrix} 2 & -1 \\ -4 & 4 \end{vmatrix} + \begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix}
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$$\begin{vmatrix} 2 & -1 \\ -4 & 4 \end{vmatrix} = 2 \cdot 4 - (-1)(-4) = 4,$$

$$\begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix} = 3 \cdot (-4) - 2 \cdot 1 = -14$$

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Home to find the inverse of a matrix A?

$$A^{-1} = \frac{1}{|A|} adj(A), \text{ where } adj(A) := (\widehat{a}_{ij})_{n \times n} \text{ with } \widehat{a}_{ij} := A_{ji}$$

Inverse of a Square Matrix

A (square) matrix A has an **inverse** A^{-1} if $A^{-1}A = AA^{-1} = I$

When is a matrix A invertible?

When the *determinant* of A is nonzero, $|A| \neq 0$

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$$EXAMPLE: \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Solving Linear Systems

To solve a linear system of n equations and n unknowns:

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\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
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If A is invertible, $\mathbf{x} = A^{-1}\mathbf{b}$

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When A is invertible. the solution to the system $Ax = b$ is

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When A is invertible. the solution to the system $Ax = b$ is

$x_j = \dfrac{\left|A_j^b\right|}{|A|}$ for every j, where A_j^b is a matrix obtained from replacing the j-th column of A by $\mathbf{b}$.

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Matrix Inversion and Elementary Transformations

Matrix **elementary transformations**

Interchanging two rows

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A^{-1} defined as $A^{-1}A = I$.

Can multiply A by successive elementary operations until it equals I.

Example: Matrix Inversion of a 2x2 Matrix

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What if A is not diagonal**?**

Diagonalisable Matrices

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P and D closely related **eigenvalues** and **eigenvectors**

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Note that eigenvectors sustain scalar multiplication

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow A(\alpha\boldsymbol{x}) = \lambda(\alpha\boldsymbol{x})$$

Finding the Eigenvalues of a Matrix

Eigenvalues of square matrix A are the *roots* of the **characteristic equation** of A

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

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Hence eigenvalues are $\lambda_1 = 1$ and $\lambda_2 (= \lambda_3) = 2$

Some Useful Eigenvalue Properties (no proof)

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$$\sum_{i=1}^{n} \lambda_i = tr(A) = \sum_{i=1}^{n} a_{ii}.$$

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For $\lambda = 1$, $A\boldsymbol{x} = \boldsymbol{x} \Leftrightarrow \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

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$$\lambda = 2,\ A\widetilde{\boldsymbol{x}} = 2\widetilde{\boldsymbol{x}} \Leftrightarrow \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix} \begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \end{pmatrix} = \begin{pmatrix} 2\widetilde{x}_1 \\ 2\widetilde{x}_2 \\ 2\widetilde{x}_3 \end{pmatrix}$$

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Eigenvectors with $\lambda = 2$ are $\widetilde{\boldsymbol{x}} = \alpha \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ with α, β not both equal to 0

Diagonalization of a Matrix

An $n \times n$ matrix A is **diagonizable** if and only if it has a set of n **linearly independent** eigenvectors $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n$, where

$$P^{-1}AP = diag(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where P is the matrix with $\boldsymbol{x}^1, \boldsymbol{x}^2, \cdots, \boldsymbol{x}^n$ as its columns, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues.

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In the previous example, we can find that

$$\begin{pmatrix} -1 & 2 & 2 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix} \begin{pmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$
$$= \; diag(1, 2, 2)$$

Diagonalization of Symmetric Matrices Results (no proof)

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1. All eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real numbers;
2. Eigenvectors associated with different eigenvalues are **orthogonal**;
3. A can be diagonalized by an **orthogonal** matrix P with columns as eigenvectors **of unit length**.

Example: Diagonalising a Symmetric Matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = 3 \text{ with } \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \\ \lambda_2 = -1 \text{ with } \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \end{cases}$$

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$$\Rightarrow \quad \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Quadratic Forms

Quadratic Form

A **quadratic form** Q of some n-vector $\boldsymbol{x}$ can be expressed in the form

$$Q(\boldsymbol{x}) := \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \boldsymbol{x}^\mathsf{T} A \boldsymbol{x},$$

for some A, the symmetric matrix associated with Q.

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$$\begin{aligned} Q(\boldsymbol{x}) &= 3x_1^2 + 0x_1x_2 + 3x_1x_3 \\ &\quad +0x_2x_1 + x_2^2 - 2x_2x_3 \\ &\quad +3x_3x_1 - 2x_3x_2 + 8x_3^2 \end{aligned}$$

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Hence the associated symmetric matrix is

$$A = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{pmatrix}$$

Definiteness of Quadratic Forms and Matrices

A quadratic form Q,with norm A, is **positive definite**, **positive semidefinite**, **negative definite**, or **negative semidefinite** accordingly if

$$Q(\boldsymbol{x}) > 0, \; Q(\boldsymbol{x}) \geq 0, \; Q(\boldsymbol{x}) < 0, \; Q(\boldsymbol{x}) \leq 0,$$

for all $\boldsymbol{x} \neq \mathbf{0}$. Q (as well as A) is **indefinite** if $Q(\boldsymbol{x}) > 0$ for some $\boldsymbol{x}$ and $Q(\boldsymbol{y}) < 0$ for some $\boldsymbol{y}$.

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Two common methods to determine definitness of a quadratic form

1. Using *eigenvalues* of A
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Likewise important for determining properties of solutions to optimization problems

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Can prove using diagonalisation of A

$$\boldsymbol{x}^\intercal A\boldsymbol{x} = \boldsymbol{x}^\intercal(PDP^\intercal)\boldsymbol{x} = \boldsymbol{y}^\intercal D\,\boldsymbol{y} = \sum_{i=1}^{n} \lambda_i y_i^2, \text{ with } \boldsymbol{y} \equiv P^\intercal\boldsymbol{x}$$

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Eigenvalues are $0, -2, -10$, hence A is *negative semidefinite*.

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The original matrix itself is also a (leading) principal minor (of order n)

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Hence A is *positive definite*

Reading

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SB Chapter 10 (not subsections 10.5-10.7) Euclidean Spaces. (Do not need to be able to prove any of the theorems, though should know the result.)