ECON30401: Time Series Econometrics

Video Exercise 2 Solution

Prediction & Seasonality

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1. $AR(2)$ process: $Y_t = \alpha + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t$ Under the i.i.d assumption

$$\mathbb{E}[\varepsilon_{T+j}|Y_T = y_T, Y_{T-1} = y_{T-1}, .., Y_1 = y_1] = \mathbb{E}[\varepsilon_{T+j}]$$

for all $j \geq 1$ where we assume $\mathbb{E}[\varepsilon_{T+j}] = 0$ which will be used to derive the predictors below. For shorthand we denote $\mathbb{E}[\cdot|Y_T = y_T, Y_{T-1} = y_{T-1}, .., Y_1 = y_1]$ by $\mathbb{E}[\cdot|Y_T = y_T]$.

a) Firstly the predictor for $T+1$

$$\begin{aligned}
\widehat{Y}_{T+1} &= \mathbb{E}[Y_{T+1}|Y_T = y_T] && \text{By definition} \\
&= \mathbb{E}[\alpha + \phi_1 Y_T + \phi_2 Y_{T-1} + \varepsilon_{T+1}|Y_T = y_T] && \text{Plug in } Y_{T+1} = \alpha + \phi_1 Y_T + \phi_2 Y_{T-1} + \varepsilon_{T+1} \\
&= \alpha + \phi_1 y_T + \phi_2 y_{T-1} && \text{By i.i.d } \mathbb{E}[\varepsilon_{T+1}|Y_T = y_T] = \mathbb{E}[\varepsilon_{t+1}] = 0
\end{aligned}$$

The predictor at $T+2$,

$$\begin{aligned}
\widehat{Y}_{T+2} &= \mathbb{E}[Y_{T+2}|Y_T = y_T] \\
&= \mathbb{E}[\alpha + \phi_1 Y_{T+1} + \phi_2 Y_T + \varepsilon_{T+2}|Y_T = y_T] && \text{Plugging in } Y_{T+2} \\
&= \alpha + \phi_1 \mathbb{E}[Y_{T+1}|Y_T = y_T] + y_T + \mathbb{E}[\epsilon_{T+2}|Y_T = y_T] \\
&= \alpha + \phi_1 \widehat{Y}_{t+1} + \phi_2 y_T && \text{As } \mathbb{E}[\varepsilon_{T+2}|Y_T = y_T] = 0 \text{ and } \hat{Y}_{T+1} = \mathbb{E}[Y_{T+1}|Y_T = y_T] \\
&= \alpha + \phi_1(\alpha + \phi_1 y_T + \phi_2 y_{T-1}) + \phi_2 y_T && \text{Plugging in } \hat{Y}_{T+1} = \alpha + \phi_1 y_T + \phi_2 y_{T-1} \\
&= \alpha(1 + \phi_1) + (\phi_1^2 + \phi_2) y_T + \phi_1 \phi_2 y_{T-1} && \text{Expanding out}
\end{aligned}$$

b) By the i.i.d assumption $Var[Y_{T+i}|Y_T = y_T] = \mathbb{E}[Y_{T+i} - \widehat{Y}_{T+i}]^2$ so that

$FE(i) = \mathbb{E}[Y_{T+i} - \widehat{Y}_{T+i}]^2$, where $\widehat{Y}_{T+i} = \mathbb{E}[Y_{T+i}|Y_T = y_T]$.

$$\begin{aligned} FE(1) &= \mathbb{E}[Y_{T+1} - \widehat{Y}_{T+1}]^2 \\ &= \mathbb{E}\left[\alpha + \phi_1 Y_T + \phi_2 Y_{T-1} + \varepsilon_{T+1} - (\alpha + \phi_1 Y_T + \phi_2 Y_{T-1})^2\right] \quad \text{Plugging in } \hat{Y}_{T+1} \\ &= \mathbb{E}[\varepsilon_{t+1}^2] = \sigma^2 \end{aligned}$$

$$\begin{aligned} FE(2) &= \mathbb{E}[Y_{t+2} - \widehat{Y}_{T+2}]^2 \\ &= \mathbb{E}\left[\alpha + \phi_1 Y_{T+1} + \phi_2 Y_T + \varepsilon_{T+2} - (\alpha + \phi_1 \widehat{Y}_{T+1} + \phi_2 Y_T)^2\right] \\ &= \mathbb{E}\left[\varepsilon_{T+2} + \phi_1 (Y_{T+1} - \widehat{Y}_{T+1})^2\right] \\ &= \mathbb{E}\left[(\varepsilon_{T+2} + \phi_1 \varepsilon_{T+1})^2\right] \quad \text{As } Y_{T+1} - \hat{Y}_{T+1} = \varepsilon_{T+1} \\ &= \sigma^2(1 + \phi_1^2) \quad \text{Standard MA(1) Variance proof} \end{aligned}$$

c) The result in (b) shows that $FE[2] > FE[1]$ when $\phi_1 \neq 0$. It is anticipated that $FE[i]$ would continue to increase with i, since the observations to time t become less relevant as forecasts further into the future are considered. Since FE is a measure of forecast accuracy, the expected accuracy declines as the forecast horizon i increases.

2(a)

$$Y_t = \alpha_0 + \alpha_1 D_{1t} + \alpha_2 D_{2t} + \alpha_3 D_{3t} + \phi_2 Y_{t-2} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

The aim is to derive the mean in each quarter

$\mu_1 = \mathbb{E}[Y_t|q = 1]$, $\mu_2 = \mathbb{E}[Y_t|q = 2]$,$\mu_3 = \mathbb{E}[Y_t|q = 3]$, $\mu_4 = \mathbb{E}[Y_t|q = 4]$.

Quarter 1$(q = 1)$ $\mu_1 = \mathbb{E}[Y_t|q = 1]$ if time t is quarter 1 $D_{1t} = 1$ and $D_{2t} = D_{3t} = 0$

$$q = 1 \qquad Y_t = \alpha_0 + \alpha_1 + \phi_2 Y_{t-2} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Taking expectations on both sides in quarter 1

$$\mathbb{E}[Y_t|q = 1] = \alpha_0 + \alpha_1 + \phi_2 \mathbb{E}[Y_{t-2}|q = 1] + \mathbb{E}[\varepsilon_t] + \theta_1 \mathbb{E}[\varepsilon_{t-1}]$$

where $\mathbb{E}[\varepsilon_t] = \mathbb{E}[\varepsilon_{t-1}] = 0$ by the WN assumption.

If time t is $q = 1$ then time $t - 2$ is quarter 3, so that $\mathbb{E}[Y_{t-1}|q = 1] = \mu_3$.

$$\mu_1 = \alpha_0 + \alpha_1 + \phi_1 \mu_3$$

Likewise we can perform the same analysis in deriving the mean in **Quarter 2**. When $q = 2$ then

$$q = 2 \qquad Y_t = \alpha_0 + \alpha_2 + \phi_2 Y_{t-2} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

$$\mathbb{E}[Y_t|q = 2] = \alpha_0 + \alpha_1 + \phi_2 \mathbb{E}[Y_{t-2}|q = 2] + \mathbb{E}[\varepsilon_t] + \theta_1 \mathbb{E}[\varepsilon_{t-1}]$$

$\mathbb{E}[Y_{t-2}|q = 2] = \mu_4$ as if time t is quarter 2 then times period $t - 2$ is quarter 4.

$$\mu_2 = \alpha_0 + \alpha_2 + \phi_2 \mu_4$$

We can perform the same analysis for quarter 3 and 4 to show

$$\mu_3 = \alpha_0 + \alpha_3 + \phi_2\mu_1$$

$$\mu_4 = \alpha_0 + \phi_1\mu_2$$

we then have 4 equations to solve for $\mu_1, \mu_2, \mu_3, \mu_4$.

Firstly we have $\mu_1 = \alpha_0 + \alpha_1 + \phi_2\mu_3$ plugging in $\mu_3 = \alpha_1 + \alpha_2 + \phi_2\mu_1$

$$\mu_1 = \alpha_0 + \alpha_1 + \phi_2(\alpha_0 + \alpha_2 + \phi_2\mu_1)$$

Solving we find

$$\mu_1 = \frac{\alpha_0 + \alpha_1 + \phi_2(\alpha_0 + \alpha_2)}{1 - \phi_2}$$

which we can plug in to solve for μ_3.

$$\mu_3 = \alpha_1 + \alpha_2 + \phi_2\left(\frac{\alpha_0 + \alpha_1 + \phi_2(\alpha_0 + \alpha_2)}{1 - \phi_2}\right)$$

Likewise we can solve for μ_2, μ_4.

2(b)

$$
\begin{aligned}
Y_t^{SA} &= Y_t - (\alpha_0 + \alpha_1 D_{1t} + \alpha_2 D_{2t} + \alpha_3 D_{3t} + \phi_2 Y_{t-2}) \\
&= \varepsilon_t + \theta_1 \varepsilon_{t-1}
\end{aligned}
$$

which is an MA(1) process.

$$
Var(Y_t^{SA}) = Var(\varepsilon_t + \theta_1 \varepsilon_{t-1}) = \sigma^2(1 + \theta_1^2)
$$