ECON5221

Video Exercise Sheet 1 Solutions

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1. A process has an $MA(\infty)$ representation

$$\begin{aligned} Y_t &= \alpha + \sum_{s=0}^{\infty} \eta_s \varepsilon_{t-s} \\ \boxed{1}\ \eta_s &= \phi_1 \eta_{s-1}, \quad s \geq 2 \\ \boxed{2}\ \eta_0 &= 1, \quad \boxed{3}\ \eta_1 = \phi_1 + \theta_1 \end{aligned}$$

where the coefficients η_s satisfy 1-3. It turns out this MA(∞) is an ARMA(1,1) process which is what part (a) asks you to show.

$\boxed{1}$ says that the MA(∞) coefficients η_s for $s \geq 2$ are a first order difference equation where

$\boxed{2}$ is the initial condition, i.e $\eta_1 = \phi_1 + \theta_1$ and

$\boxed{3}$ $\eta_0 = 1$ which is convention with any MA(∞) process.

a)

$$\begin{aligned} Y_t &= \alpha \quad + \varepsilon_t \quad + \eta_1 \varepsilon_{t-1} \quad + \quad \eta_2 \varepsilon_{t-2} \quad + \quad \eta_3 \varepsilon_{t-3} \quad + \quad \ldots \qquad \boxed{\text{Expanding out } Y_t = \alpha + \sum_{s=0}^{\infty} \eta_s \varepsilon_{t-s}} \\ Y_{t-1} &= \alpha \qquad\qquad + \quad \varepsilon_{t-1} \quad + \quad \eta_1 \varepsilon_{t-2} \quad + \quad \eta_2 \varepsilon_{t-3} \quad + \quad \ldots \qquad \boxed{\text{Expanding out } Y_{t-1} = \alpha + \sum_{s=0}^{\infty} \eta_s \varepsilon_{t-s-1}} \end{aligned}$$

Then subtact $\phi_1 Y_{t-1}$ from Y_t above and we find

$$\begin{aligned} Y_t - \phi_1 Y_{t-1} &= \underbrace{(\alpha - \phi_1 \alpha)}_{\boxed{A}} + \varepsilon_t + \underbrace{(\eta_1 - \phi_1)\varepsilon_{t-1}}_{\boxed{B}} + \underbrace{(\eta_2 - \phi_1 \eta_1)\varepsilon_{t-2} + (\eta_3 - \phi_1 \eta_2)\varepsilon_{t-3} + \ldots}_{\boxed{C}} \\ &= \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} \end{aligned}$$

$\boxed{A}$ Defining $\mu = \alpha(1 - \phi_1)$

$\boxed{B}$ By (3) $\eta_1 = \phi_1 + \theta_1$ hence $\eta_1 - \phi_1 = \theta_1$.

$\boxed{C}$ By (2) $\eta_s - \phi_1 \eta_{s-1} = 0$ for $s \geq 2$, i.e $\eta_2 - \phi_1 \eta_1 = 0$, $\eta_3 - \phi_1 \eta_2 = 0$ ....

Thus, Y_t [the MA(∞) process above with coefficients satisfying 1-3] is the $ARMA(1,1)$ process

$$Y_t = \alpha + \phi_1 Y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}.$$

b) Successively substituting in $\eta_s = \phi_1 \eta_{s-1}$, $s \geq 2$, then

$$\begin{aligned} \eta_s &= \phi_1 \eta_{s-1} \\ &= \phi_1^2 \eta_{s-2} && \text{Substituting } \eta_{s-1} = \phi_1 \eta_{s-2} \\ &= \phi_1^3 \eta_{s-3} && \text{Substituting } \eta_{s-2} = \phi_1 \eta_{s-3} \\ &\vdots \quad \text{Recursing bank (s-2) times} \\ &= \phi_1^{s-1} \eta_1, \; s \geq 2 \end{aligned}$$

The starting values given in (2) $\eta_1 = \phi_1 + \theta_1$.

$$\eta_s = \phi_1^{s-1}(\phi_1 + \theta_1) \; s \geq 2$$

Also since $\eta_1 = \phi_1 + \theta_1 = \phi_1^0(\phi_1 + \theta_1)$

$$\eta_s = \phi_1^{s-1}(\phi_1 + \theta_1) \; s \geq 1$$

c) The **Absolute Summablity Condition** is

$$\sum_{s=0}^{\infty} |\eta_s| < \infty$$

For this process

$$\begin{aligned} \sum_{s=0}^{\infty} |\eta_s| &= 1 + \sum_{s=1}^{\infty} |\eta_s| && \text{Breaking up the sum and using } \eta_0 = 1 \\ &= 1 + \sum_{s=1}^{\infty} \left| \phi_1^{s-1}(\phi_1 + \theta_1) \right| && \text{Substituting } \eta_s = \phi_1^{s-1}(\phi_1 + \theta_1) \text{ for } s \geq 1 \\ &= 1 + \sum_{s=1}^{\infty} |\phi_1|^{s-1} |\phi_1 + \theta_1| && \text{As } |ab| = |a||b| \\ &= 1 + |\phi_1 + \theta_1| \sum_{s=1}^{\infty} |\phi_1|^{s-1} && \text{Taking } |\phi_1 + \theta_1| \text{ Outside of the sum} \\ &= 1 + |\phi_1 + \theta_1| \sum_{s=0}^{\infty} |\phi_1|^{s} && \text{Geometric Progression in } |\phi_1| \\ &= 1 + \frac{|\phi_1 + \theta_1|}{1 - |\phi_1|} && \text{If } |\phi_1| < 1 \end{aligned}$$

Hence $|\phi_1| < 1$ then $\sum_{s=0}^{\infty} |\eta_s|$ is finite. If $|\phi_1| \geq 1$ then it is clear from above that $\sum_{s=0}^{\infty} |\eta_s|$ is infinite.

Therefore, the $ARMA(1,1)$ is stationary when

$$|\phi_1| < 1.$$

d) Under the assumptions the $ARMA(1,1)$ process is stationary i.e $|\phi_1| < 1$ (hence the variance is finite and exists). using (1),

$$\begin{aligned}
Var[Y_t] &= \sigma^2 \sum_{s=0}^{\infty} \eta_s^2 && \text{General formula for variance of } MA(\infty) \\
&= \sigma^2 \left\{ 1 + \sum_{s=1}^{\infty} \eta_s^2 \right\} && \text{Breaking up the sum and using } \eta_0 = 1 \\
&= \sigma^2 \left\{ 1 + (\theta_1 + \phi_1)^2 \sum_{s=1}^{\infty} \phi_1^{2(s-1)} \right\} && \text{Subst. } \eta_s = (\theta_1 + \phi_1)\phi_1^{s-1} \text{ for } s \geq 1 \\
&= \sigma^2 \left\{ 1 + (\phi_1 + \theta_1)^2 \sum_{s=0}^{\infty} \phi_1^{2s} \right\} && \text{Re-expressing the sum} \\
&= \sigma^2 \left\{ 1 + \frac{(\phi_1 + \theta_1)^2}{1 - \phi_1^2} \right\} && \text{Geometric Progession in } \phi_1^2 \text{ where } \phi_1^2 < 1 \\
&= \sigma^2 \frac{1 - \phi_1^2 + \phi_1^2 + \theta_1^2 + 2\phi_1\theta_1}{1 - \phi_1^2} \\
&= \sigma^2 \frac{(1 + 2\phi_1\theta_1 + \theta_1^2)}{1 - \phi_1^2}
\end{aligned}$$

e) We may use the general formula for the auto-covariance function of a stationary MA(∞)

$$\gamma(k) = \sigma^2 \textstyle\sum_{s=0}^{\infty} \eta_s \eta_{s+k} \qquad k = 0, 1, 2....$$

. Firstly for $k = 1$, i.e $Cov[Y_t, Y_{t-1}]$

$$\begin{aligned}
\gamma(1) &= \sigma^2 \left(\sum_{s=0}^{\infty} \eta_s \eta_{s+1} \right) \\
&= \sigma^2 \left(\eta_1 + \sum_{s=1}^{\infty} \eta_s \eta_{s+1} \right) && \text{Breaking up the sum where } \eta_0 = 1 \\
&= \sigma^2 \left((\phi_1 + \theta_1) + (\phi_1 + \theta_1)^2 \sum_{s=1}^{\infty} \phi_1^{2s-1} \right) && \text{Subst. } \eta_s = (\theta_1 + \phi_1)\phi_1^{s-1} \text{ for } s \geq 1 \\
&= \sigma^2 \left((\phi_1 + \theta_1) + (\phi_1 + \theta_1)^2 \phi_1 \sum_{s=0}^{\infty} \phi_1^{2s} \right) && \text{Re-expressing the sum; } \phi_1^{2s-1} = \phi_1^{2(s-1)}\phi_1 \\
&= \sigma^2 \left((\phi_1 + \theta_1) + \frac{(\phi_1 + \theta_1)^2 \phi_1}{1 - \phi_1^2} \right) && \text{Geometric Progession in } \phi_1^2 \text{ where } \phi_1^2 < 1 \\
&= \sigma^2 (\phi_1 + \theta_1) \left(1 + \frac{(\phi_1 + \theta_1)\phi_1}{1 - \phi_1^2} \right)
\end{aligned}$$

Now we solve for $\gamma(k)$ for $k \geq 2$

$$\begin{aligned}
\gamma(k) &= \sigma^2 \sum_{s=0}^{\infty} \eta_s \eta_{s+k} \\
&= \sigma^2 \left(\eta_k + \sum_{s=1}^{\infty} \eta_s \eta_{s+k} \right) && \text{Breaking up the sum where } \eta_0 = 1 \\
&= \sigma^2 \left(\phi_1^{k-1}(\phi_1 + \theta_1) + (\phi_1 + \theta_1)^2 \sum_{s=1}^{\infty} \phi_1^{2s+k-2} \right) && \text{Subst. } \eta_s = (\theta_1 + \phi_1)\phi_1^{s-1} \text{ for } s \geq 1 \\
&= \sigma^2 \left(\phi_1^{k-1}(\phi_1 + \theta_1) + \phi_1^{k-1}(\phi_1 + \theta_1)^2 \sum_{s=1}^{\infty} \phi_1^{2s-1} \right) && \text{Noting } \phi_1^{2s+k-2} = \phi_1^{2s-1}\phi_1^{k-1} \\
&= \sigma^2 \left(\phi_1^{k-1}(\phi_1 + \theta_1) + \phi_1^{k-1}(\phi_1 + \theta_1)^2 \phi_1 \sum_{s=0}^{\infty} \phi_1^{2s} \right) && \text{Re-expressing the sum; } \phi_1^{2s-1} = \phi_1^{2(s-1)}\phi_1 \\
&= \sigma^2 \phi_1^{k-1} \left[(\phi_1 + \theta_1) + (\phi_1 + \theta_1)^2 \phi_1 \sum_{s=0}^{\infty} \phi_1^{2s} \right] && \gamma(1) = (\phi_1 + \theta_1) + (\phi_1 + \theta_1)^2 \phi_1 \textstyle\sum_{s=0}^{\infty} \phi_1^{2s} \\
&= \phi_1^{k-1} \gamma(1)
\end{aligned}$$

Hence we have shown

$$\gamma(k) = \begin{cases} \sigma^2(\phi_1 + \theta_1)\left[1 + \frac{(\phi_1+\theta_1)\phi_1}{1-\phi_1^2}\right] & : k = 1 \\ \phi_1^{k-1}\gamma(1) & : k \geq 2 \end{cases}$$

By definition $\rho(k) = \gamma(k)/\gamma(0)$

$$\begin{aligned}
\rho(1) &= \frac{\sigma^2(\phi_1 + \theta_1)\left[1 + \frac{(\phi_1+\theta_1)\phi_1}{1-\phi_1^2}\right]}{\sigma^2 \frac{(1+2\phi_1\theta_1+\theta_1^2)}{1-\phi_1^2}} \\
&= \frac{(\phi_1 + \theta_1)\left[1 - \phi_1^2 + (\phi_1 + \theta_1)\phi_1\right]}{(1 + 2\phi_1\theta_1 + \theta_1^2)} && \text{Multiply top and bottom by } (1 - \phi_1^2) \\
&= \frac{(\phi_1 + \theta_1)\left[1 + \theta_1\phi_1\right]}{(1 + 2\phi_1\theta_1 + \theta_1^2)}
\end{aligned}$$

Then for $k \geq 2$ $\rho(k) = \phi_1^{k-1}\frac{\gamma(1)}{\gamma(0)} = \phi_1^{k-1}\rho(1)$ by definition as $\rho(1) = \gamma(1)/\gamma(0)$.

Putting it all together

$$\rho(k) = \begin{cases} \frac{(\phi_1+\theta_1)(1+\phi_1\theta_1)}{1+2\phi_1\theta_1+\theta_1^2} & : k = 1 \\ \phi_1^{k-1}\rho(1) & : k \geq 2 \end{cases}$$

f) If $\theta_1 = -\phi_1$ then $\rho(1) = 0$ as $\rho(1) = \frac{(\phi_1+\theta_1)(1+\phi_1\theta_1)}{1+2\phi_1\theta_1+\theta_1^2}$ then $\rho(k) = 0$ for $k \geq 1$ as $\rho(k) = \phi_1^{k-1}\rho(1)$. Hence the process is uncorrelated at all lags.

To see why note for the ARMA(1,1) when $\theta_1 = -\phi_1$

$$\begin{aligned}
Y_t &= \mu + \phi_1 Y_{t-1} - \phi_1 \varepsilon_{t-1} + \varepsilon_t \qquad \text{ARMA(1,1) with } \theta_1 = -\phi_1 \\
&= \mu + \phi_1 (Y_{t-1} - \varepsilon_{t-1}) + \varepsilon_t \\
&= \mu + \phi_1 \mu + \phi_1^2 (Y_{t-2} - \varepsilon_{t-2})) + \varepsilon_t \qquad \text{Subst. } Y_{t-1} = \mu + \phi_1 (Y_{t-2} - \varepsilon_{t-2}) + \varepsilon_{t-1} \\
&\vdots \qquad \text{Recuring back j periods} \\
&= \mu + \mu\phi_1 + .. + \mu\phi_1^j + \phi_1^{j+1} (Y_{t-j-1} - \varepsilon_{t-j-1}) + \varepsilon_t \\
&\vdots \qquad \phi_1^{j+1} (Y_{t-j-1} - \varepsilon_{t-j-1}) \to 0 \text{ as } j \to \infty \text{ as } |\phi_1| < 1 \\
&= \frac{\mu}{1 - \phi_1} + \varepsilon_t
\end{aligned}$$

so that $Y_t = \frac{\mu}{1-\phi_1} + \varepsilon_t$ and hence why Y_t is uncorrelated in all time periods for an ARMA(1,1) with $\phi_1 = -\theta_1$ and $|\phi_1| < 1$.