Econometric Time Series Analysis|EC5221

Wk2: Stationary Univariate Time Series - Estimation and Inference

Nicky Grant (Semester 2, 2020/2021)

Table of contents

Recap

Distinction between **process** $\{Y_t\}$ and **sample realisation** $\{y_t\}$

White noise & **stationarity**

Measuring statistical dependence- **auto-correlation function [ACF]**

Moments and stationarity conditions of **ARMA** processes

Focused on theoretical properties of ARMA processes assuming model parameters **known**

What We Will Cover

Sample moments & asymptotic properties

Quick refresher on convergence in probability and distribution

Maximum likelihood estimation of ARMA models

Sample Moments

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E.g. How to form a consistent estimator of the population mean $\mathbb{E}[Y_t]$?

i.e. an estimator **converging in probability** to $\mathbb{E}[Y_t]$ as $T \to \infty$

Refresher: Convergence in Probability and Distribution

Definition: Convergence in Probability

A stochastic sequence X_T on $T = 1, 2, \cdots$ converges in probability to X if $\forall\ \epsilon > 0$

$$\Pr\{|X_T - X| < \epsilon\} \underset{T \to \infty}{\to} 1$$

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Shorthand: $X_T \xrightarrow{p} X$ and $X_T \xrightarrow{d} X$ respectively

Time Averages

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A crucial assumption for consistency is the process generating the sample is stationary

Sample Mean - Asymptotic Properties

Time average in sample $\bar{y}_T := \frac{1}{T}\sum_{t=1}^{T} y_t$ drawn from the r.v. $\bar{Y}_T := \frac{1}{T}\sum_{t=1}^{T} Y_t$

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Theorem 1.5: Consistency of Sample Mean for Serial Dependent Processes

Any weakly-stationary process $\{Y_t\}$ with absolute summable covariances

$$\bar{Y}_T \xrightarrow{p} \mu$$

Exercise Prove the above result [*Hint: use Chebychev's Inequality*]

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Theorem 1.8: Central Limit Theorem for Serially Dependent Processes

If $Y_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$ where $\{\varepsilon_t\}$ is i.i.d, $\mathbb{E}[\varepsilon_t] = 0$, $\mathbb{E}[\varepsilon_t^2] < \infty$ and $\sum_{j=0}^{\infty} |\psi_j| < \infty$

$$\sqrt{T}(\bar{Y}_T - \mu) \xrightarrow{d} N\left(0, \sum_{j=-\infty}^{\infty} \gamma(j)\right)$$

Sample Second Order Moments

Definition: Sample Auto-Covariance Function

$$\hat{\gamma}_T(k) = \frac{1}{T-k} \sum_{t=k+1}^{T} \left(Y_t - \bar{Y}_T\right)\left(Y_{t-k} - \bar{Y}_T\right) \qquad \forall \;\; 0 \leq k < T$$

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$\hat{\gamma}_T(k)$ and $\hat{\rho}_T(k)$ evaluated at sample realisation $(y_1, \cdots, y_T)'$ to form **sample estimate**

Asymptotic Properties of Second Order Sample Moments

For linear processes $Y_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$ where $\{\varepsilon_t\}$ is i.i.d and $\sum_{j=0}^{\infty} |\psi_j| < \infty$

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Theorem 1.11: Consistency of Sample Auto-covariances for Stationary Processes

If $E[|\varepsilon_t|^r] < \infty$ for some $r > 2$

$$\hat{\gamma}_T(k) \xrightarrow{p} \gamma(k)$$

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Theorem 1.13: Consistency of Sample Auto-covariances for Stationary Processes

If $E[|\varepsilon_t|^r] < \infty$ for some $r > 2$

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Proven in **H** pp.192-193 **[Not Examinable]**

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Theorem 1.15: Consistency of Sample Auto-covariances for Stationary Processes

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Proven in **H** pp.192-193 **[Not Examinable]**

Theorem 1.16: Limit Distn. of Sample Auto-covariances for Stationary Processes

If $E[\varepsilon_t^4] < \infty$

$$\sqrt{T}(\hat{\gamma}_T(k) - \gamma(k)) \xrightarrow{d} N\left(0, \sum_{s=0}^{\infty} (\rho(s+k) + \rho(s-k) - \rho(s)\rho(k))^2\right)$$

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Theorem 1.17: Consistency of Sample Auto-covariances for Stationary Processes

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Theorem 1.18: Limit Distn. of Sample Auto-covariances for Stationary Processes

If $E[\varepsilon_t^4] < \infty$

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We do **not** prove Theorem 1.18

Ergodicity

Definition: Ergodicity (mean)

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Stationary process ergodic for mean if $\sum_{j=0}^{\infty} |\gamma(j)| < \infty$ (follows from Theorem 1.7)

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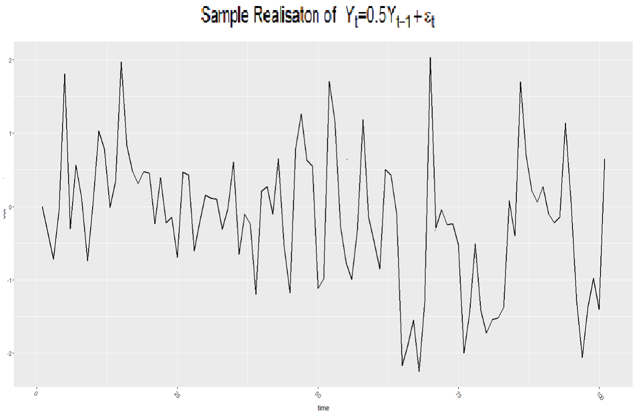
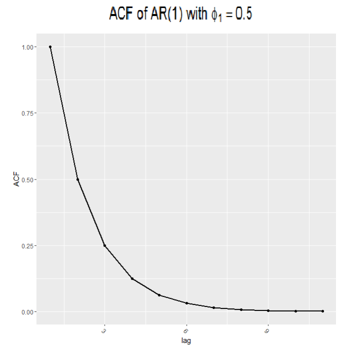
Ergodicity & stationarity closely linked- but a process can be stationary & not ergodic (see **H** p. 47)

Statistical Properties of Sample ACF

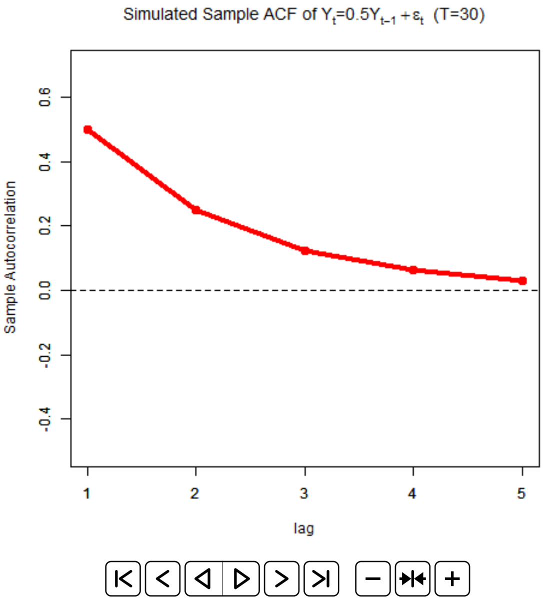
Simulation Example: Properties of Sample Autocorrelation of AR(1)

Suppose the true process is $Y_t = 0.5Y_{t-1} + \varepsilon_t \quad \varepsilon_t \overset{i.i.d}{\sim}$ N(0,1)





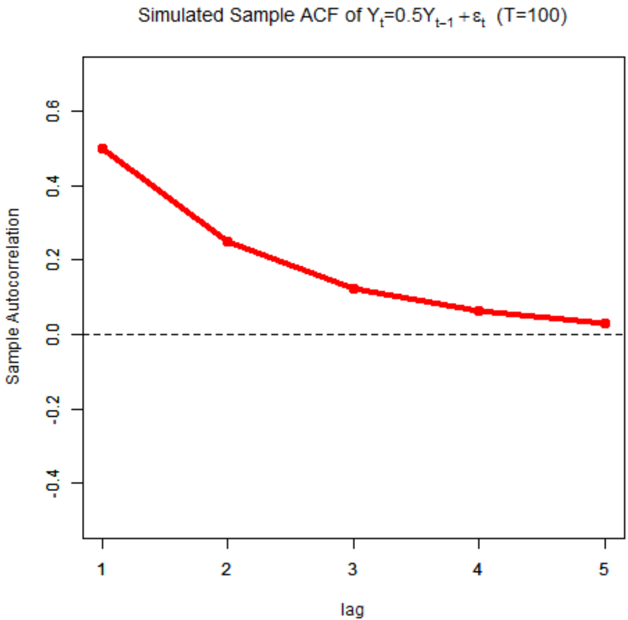
By Theorems 1.7 $\hat{\rho}_T(k) \xrightarrow{p} \rho(k)$



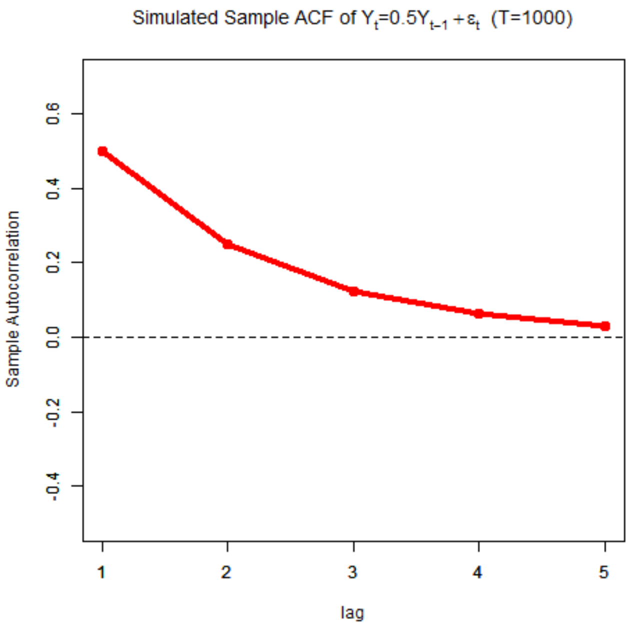
1. Generate a sample $\{y_1^b, .., y_T^b\}$ from $Y_t = 0.5Y_{t-1} + \varepsilon_t$ where $\varepsilon_t \overset{i.i.d}{\sim} N(0,1)$ for $T = 30$

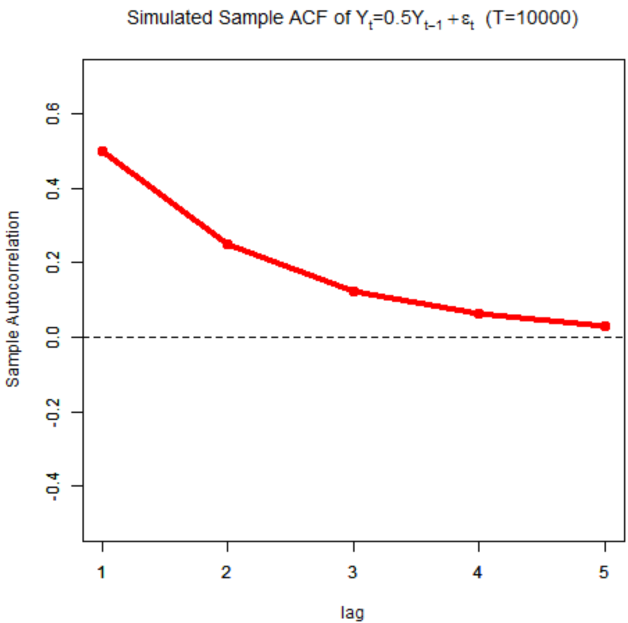
2. Calculate $\hat{\rho}_T^b(1)$,..,$\hat{\rho}_T^b(1)$ for each sample b

3. Plot Sample ACF $\hat{\rho}_T^b(1)$,..,$\hat{\rho}_T^b(5)$ for $b = \{1, .., 100\}$.

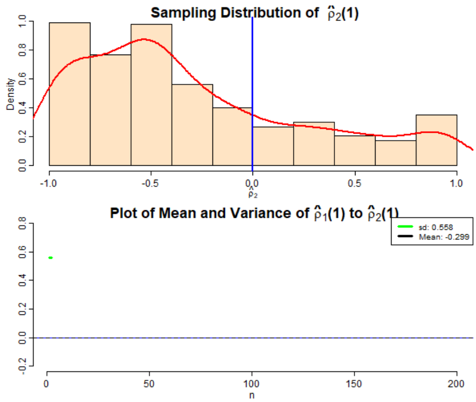


Simulated Sample ACF of Y_t=0.5Y_{t-1} + ε_t (T=100)
Sample Autocorrelation
-0.4
-0.2
0.0
0.2
0.4
0.6
1
2
3
4
5
lag


Simulated Sample ACF of Yt=0.5Yt-1 + εt (T=1000)
Sample Autocorrelation
0.6
0.4
0.2
0.0
-0.2
-0.4
1
2
3
4
5
lag


Simulated Sample ACF of Y_t=0.5Y_{t-1}+ε_t (T=10000)
Sample Autocorrelation
lag
-0.4
-0.2
0.0
0.2
0.4
0.6
1
2
3
4
5

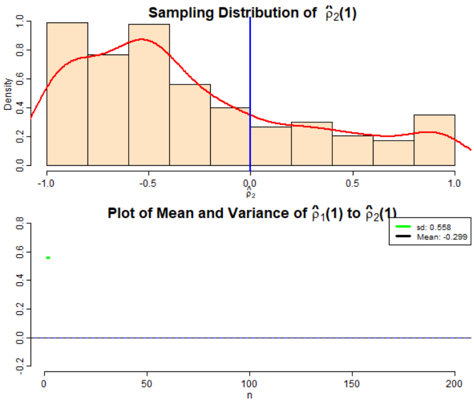
Simulation: Sampling Distribution of Sample Autocorrelation



Consider $Y_t = \varepsilon_t$ [i.e. $\phi_1 = 1$]

Theorem 1.8 $\Rightarrow \hat{\rho}_T(1) \overset{a}{\sim} N(0, 1/T)$

Sampling Distribution of $\hat{\rho}_T(1)$ for $T = 2, .., 200$



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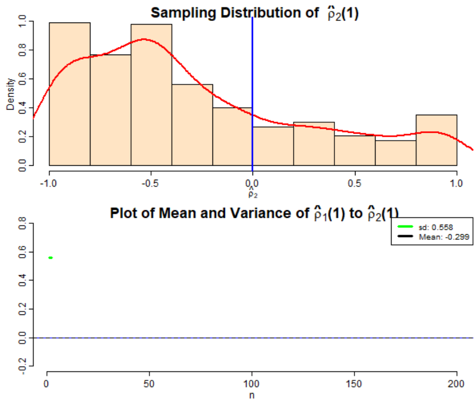
Simulation: Sampling Distribution of Sample Autocorrelation



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2. Calculate $\hat{\rho}_T^b(1)$ for samples $b = \{1, .., 1000\}$

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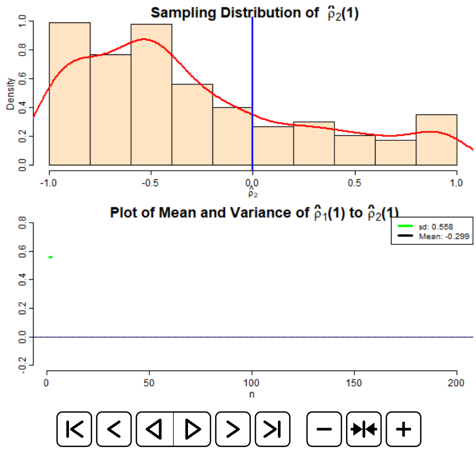

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2. Calculate $\hat{\rho}_T^b(1)$ for samples $b = \{1, .., 1000\}$
3. Plot distribution of $\hat{\rho}_T^b(1)$ for $b = \{1, .., 1000\}$

Maximum Likelihood Estimation of ARMA Processes

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$h_{Y_T,\cdots,Y_1}(c_T, \cdots, c_1) \equiv f_{Y_T,\cdots,Y_1}(c_T, \cdots, c_1; \theta)$ for some $\theta = \theta_0 \in \Theta$

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If θ_0 unique, $\{Y_t\}$ strictly stationary (+ regularity conditions)

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$$\hat{\theta}_{ML} \xrightarrow{p} \theta_0$$

$$\sqrt{T}(\hat{\theta}_{ML} - \theta_0) \xrightarrow{d} N(0, I(\theta_0)^{-1}) \quad \text{where } I(\theta) := -\mathbb{E}\left[\frac{\partial^2}{\partial\theta\partial\theta'} \ln L(\theta)\right]$$

Deriving Likelihood Function for Dependent Process

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T=2 $\quad f_{Y_2,Y_1}(y_2, y_1; \theta) = f_{Y_2|Y_1}(y_2|y_1; \theta) f_{Y_1}(y_1; \theta)$

Deriving Likelihood Function for Dependent Process

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T=3 $\quad f_{Y_3,Y_2,Y_1}(y_3, y_2, y_1; \theta) = f_{Y_3|Y_2,Y_1}(y_3|y_2, y_1; \theta) f_{Y_2|Y_1}(y_2|y_1; \theta) f_{Y_1}(y_1; \theta)$

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ML Estimation of Stationary Gaussian AR(1) Process

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Use joint density of vector Gaussian variable [*Do as an exercise*, see **H** pp. 119-122]

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Exact ML requires knowledge of whole joint density fn. of $(Y_1, \cdots, Y_T)'$ up to some parameter θ

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See **H** pp. 123-125 for full derivation of log-likelihood fn. of Gaussian AR(p)

Full derivation of exact ML log-likelihood fn. of Gaussian AR(p) **not** examinable for $p > 2$

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Conditional ML estimator of $(c, \phi_1, \cdots, \phi_p)'$ equivalent to **ordinary least squares**

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Not the case for exact ML estimator (inconsistent)

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Derive likelihood function similar to the AR(1) using Bayes Rule

Difference with AR(1): the 'regressor' ϵ_{t-1} is **not observed**

Conditional ML - Gaussian MA(1) Process

Gaussian MA(1): $Y_t = \alpha + \theta_1 \varepsilon_{t-1} + \varepsilon_t$ & $\varepsilon_t \overset{i.i.d}{\sim} N(0, \sigma^2)$

True Parameter: $\theta = \theta_0 = (\alpha_0, \theta_{10}, \sigma_0^2)$ **unknown**

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Express **realised shocks** ε_t as function of **observable** $(y_{t-1},\cdots,y_1)$

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Conditional likelihood:

$$f_{Y_T,\cdots,Y_1|\varepsilon_0=0}(y_T,\cdots,y_1,;\theta) = f_{Y_1|\epsilon_0=0}(y_1;\theta)\prod_{t=2}^{T} f_{Y_t|Y_{t-1},\cdots,Y_1,\varepsilon_0=0}(y_t|y_{t-1},\cdots,y_1;\theta)$$

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$$\varepsilon_1 = y_1 - \alpha \qquad [Y_1 = y_1]$$

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Conditional on $\varepsilon_0 = 0$ can express realised shocks ε_t as function of $(y_{t-1},\cdots,y_1)$

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$$\varepsilon_2 = y_2 - \alpha - \theta_1\varepsilon_1 \qquad [Y_1 = y_1, Y_2 = y_2]$$

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$Y_1|\varepsilon_0 = 0 \sim N(\alpha,\sigma^2)$

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$$\vdots$$

$$\varepsilon_T = y_T - \alpha - \theta_1\varepsilon_{T-1} \qquad [Y_1 = y_1,\cdots,Y_T = y_T]$$

$Y_1|\varepsilon_0 = 0 \sim N(\alpha,\sigma^2)$

$(Y_t|Y_{t-1} = y_{t-1},\cdots Y_1 = y_1,\varepsilon_0 = 0)$ same distribution as $(Y_t|\varepsilon_{t-1})$ for $t = 2,..,T$

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i.e. $(Y_t|Y_{t-1} = y_{t-1},\cdots,Y_1 = y_1,\varepsilon_0 = 0) \sim N(\alpha + \theta_1\varepsilon_{t-1},\sigma^2)$

ML Estimation of ARMA Models (what you need to know)

We will **not** cover exact ML estimation of models with MA component or with AR(p) for $p > 2$

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See **H** p.134 for conditional ML for ARMA(p,q)

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See **H** p.130 for conditional ML for MA(q)

See **H** p.134 for conditional ML for ARMA(p,q)

H pp. 132-139 for ML estimator computation in non-linear models (useful, but **not examinable**)

Reading list

Main Reading

Hamilton

[4.8] Wold's Decomposition (Lecture 1 readinf) and section on sample autocorrelations.

[5.1] Intro to ML. It may be useful to refresh your knowledge of ML estimation from your PG/UG notes.
[5.2] Conditional and exact ML estimation of Gaussian AR(1). Understand two alternative ways of deriving the log-likelihood function. You should be comfortable understanding both methods.
[5.3] Generalises 5.2 to AR(p). You need to understand all of this sub-chapter, aside from deriving the exact log-likelihood of AR(p) for p¿2
[5.4] Conditional ML estimation of MA(1) (read the part on exact ML, not examinable)
[5.5] Extends 5.4 to general MA(q), again only the conditional ML part is examinable.
[5.6] Conditional ARMA(p,q)
[5.7 Numerical optimisation (read over this material [omit the DFP if you wish], it is not examinable for my half of the course)
[5.8] ML estimation and inference. We did not cover this in lecture, a lot of this you saw in EC . You need to be comfortable doing the likelihood ratio and score test and deriving asymptotic standard errors of ML estimator.

[7.1] Review of asymptotic distribution theory
[7.2] Limit theorems for serially dependent processes pp.188-190 and CLT for stationary processes on p.195

Moving On

Covered univariate models and ML estimation of parameters of stationary processes

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Lects 3-5 extend these results to **multivariate** time series

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Study vector autoregression (**VAR**) and **impulse response functions**

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Estimation/inference from VARs (includes univariate as a special case)