Econometric Time Series Analysis|EC5221

Wk1: Stationary Univariate Time Series - ARMA Processes and Stationarity

Nicky Grant (Semester 2, 2020/2021)

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Course Details

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Time series econometrics- tool to analyse economic time series data

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Office Hours:

Nicky:Thu 1-3pm (or by appointment) [F14]

Lecture Topics

1. Stationary Univariate Time Series - **ARMA Processes and Stationarity**
2. Stationary Univariate Time Series - **Estimation and Inference**

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6. Univariate Non-stationary Time Series
7. Multivariate Time Series: Unit roots and Co-integration
8. Time Series Models of Heteroskedasticity
9. Introduction to Continuous Time Econometric Models
10. Likelihood methods for estimating continuous time models with discrete data
11. Estimating volatility in the presence of microstructure noise

Assessment and Reading

25% Continuous Assessment

Class tests **wk7** and **wk11**

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75% Final Examination in May

3 hours to answer *3 out of 5 questions*

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Lecture notes accompany some of the course

See Course Outline and lecture materials for core/further reading

Introduction to Time Series

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Focus on **discrete** time series in first half of course

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US Quarterly GDP 1980-2020 ($bn)

Figure 1: Source: https://fred.stlouisfed.org/series/GDP

FTSE100 Stock Return 1994-2020 (Monthly)

Figure 2: Source: https://finance.yahoo.com/quote/%5EFTSE

US GDP Growth 1980-2020 (Quarterly)

Figure 3: Source: https://fred.stlouisfed.org/series/GDP

UK/GER Consumer Price Inflation 1990-2020 (Quarterly)

Figure 4: Source: https://fred.stlouisfed.org/series/GBRCPIALLMINMEI

Fundamental Statistical Definitions

Distribution of a Time Series Process

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We observe **one realisation**, the sample data $(y_1, \cdots, y_T)'$

Definition: Population Moments of Y_t

For any function $g(\cdot)$

$$\mathbb{E}[g(Y_t)] := \int_{-\infty}^{\infty} g(y) f_t(y) dy$$

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Var of Y_t $\quad \mathbb{E}[(Y_t - \mu_t)^2]$ $\qquad \mathrm{Var}(Y_t) := \int_{-\infty}^{\infty} (y - \mu_t)^2 f_t(y) dy$

Moments of Time Series Process (bivariate)

Consider $\mathcal{T} = \{t_1, t_2\}$ where $t_1 \neq t_2 \in \mathcal{T}$

Definition: Population Moments of (Y_{t_1}, Y_{t_2})

$$\mathbb{E}[g(Y_{t_1}, Y_{t_2})] := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(y_{t_1}, y_{t_2}) f_{\mathcal{T}}(y_{t_1}, y_{t_2}) dy_{t_1} dy_{t_2}$$

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$$\text{Cov}(Y_{t_1}, Y_{t_2}) := \mathbb{E}[(Y_{t_1} - \mu_{t_1})(Y_{t_2} - \mu_{t_2})] \qquad \textbf{Covariance } Y_{t_1}, Y_{t_2}$$

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Definition: Weakly Stationary Process

A process $\{Z_t\}$ is **weakly stationary** if

$$\begin{aligned}
\mathbb{E}[Z_t] &= \mu && \text{for all } t \in \mathcal{T}, \\
\text{Var}(Z_t) &= \sigma^2 < \infty && \text{for all } t \in \mathcal{T}, \\
\text{Cov}(Z_{t_1}, Z_{t_2}) &= \gamma(|t_1 - t_2|) && \text{for all } t_1, t_2 \in \mathcal{T}.
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Definition: Strictly Stationary Process

A process $\{Z_t\}$ is **strictly stationary** if for all $t_1, \cdots t_k \in \mathcal{T}$ (for any positive integer k) the joint density of $(Z_{t_1+\tau}, ..., Z_{t_k+\tau})'$ is **invariant** to τ.

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Process generating sample data may not be stationary, e.g.

- Mean and/or (co-) variance of Z_t is not constant over time
- Z_t has a unit root (e.g. a random walk)

White Noise Process

Definition: White Noise

A process $\{Z_t\}$ is **white noise** if

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i.e. **zero mean, constant variance and no correlation across time**

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Note: white noise $\Rrightarrow$ weak stationarity (not the reverse)

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Correlations in sample data should reflect the correlation in $\{Y_t\}$ (for large T)

ARMA(p,q) Processes

Moving Average Process

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- Known as *moving average process (order 1)*

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$$\vdots$$
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Autoregressive Moving Average Processes

Definition: Autoregressive process order p

AR(p) $$Y_t = \alpha + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \varepsilon_t$$

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$p = 0$ is **MA(q)**

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Lecture 1-5 focus on processes $\{Y_t\}$ stationary

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Stationary if MA(∞) coefficients squared summable

Importance of MA(∞): Wold Decomposition

Theorem: Wold Decomposition

Any stationary process can be represented as $MA(\infty)$:

$$Y_t = \alpha + \sum_{s=0}^{\infty} \theta_s \varepsilon_{t-s}, \quad \theta_0 = 1 \qquad \varepsilon_t \sim WN(\sigma^2)$$

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Can derive mean, variance and covariances of a stationary process by finding its MA(∞) form and using general formulas for mean, variance and covariance of an MA(∞)

We use this method for the stationary AR(1) process (see **Lecture Notes 1**)

Moments ARMA Processes and Stationarity Conditions

Moments of MA(1) Process

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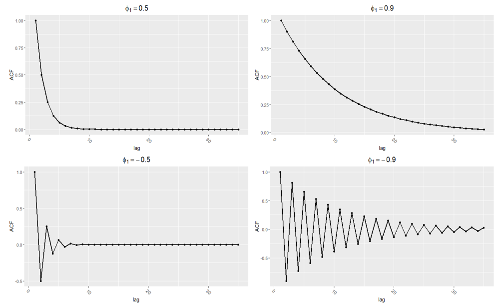

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Lecture Notes 1 & (**H** pp. 50-51) derive MA(q) autocovariance fn.

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MA(q) always stationary - *models with an AR component may not be*

A useful tool to study properties of ARMA(p,q) is the *lag operator*...

Lag Operator

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The Lag Operator L satisfies

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Intuition for Stationarity Condition in AR(p) for p=2

$$Y_t = \alpha + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \quad \Leftrightarrow \quad (1 - \phi_1 L - \phi_2 L^2) Y_t = \alpha + \varepsilon_t$$

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$$Y_t = (1 - \lambda_1 L)^{-1} (1 - \lambda_2 L)^{-1} (\alpha + \varepsilon_t)$$

Require $|\lambda_1| < 1$ **and** $|\lambda_2| < 1$ for **stationarity**

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Suppose not and $\lambda_2 = 1$

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Unbounded variance (and mean when $\alpha \neq 0$)

Stationarity Condition in General AR(p)

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$$(1 - \phi_1 L - \cdots - \phi_p L^p) Y_t = (1 - \lambda_1 L) \times \cdots \times (1 - \lambda_p) Y_t$$

Stationarity Condition in General AR(p)

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$|\lambda_j| < 1$ for $j = \{1, \cdots, p\}$ for **stationarity**

Main Reading

Hamilton

[1] Recap of difference equations and some useful (matrix) algebra. (Omit part on pth order diff. equations with repeated roots (p. 18/19). Appendix 1A optional)

[2.1] Realisation of a time series process and the lag operator

[2.2] Recursion using lag operator for first order difference series

[2.3] Second order difference series (useful for studying properties of AR(2) models)

[3.1] Expectations and stochastic processes (gives more detail on what we discussed at the start of lecture)

[3.2] White noise processes

[3.3] Moving average processes (derives mean, variance and autocovariance/correlation functions of MA(q) and MA(∞) models and discussion of the absolute/squared summability conditions.)

[3.4] Properties and stationarity conditions for AR(p) models

[3.5] ARMA processes

[3.7] Invertibility of MA processes (we did not have time to cover this in lecture but it is examinable and I shall make a video clip discussing this)

Lecture Notes 1 (available on Moodle)

Next Week

Next Week- Estimation and Inference

Sample estimation of first/second moments of a process

Study their asymptotic (large sample) properties

Maximum likelihood estimation of ARMA models and testing