ECON61001: Econometric Methods

Exercise 5: Solutions

Nicky Grant (2015)

Class Solutions- Diagnostic Testing and Robust Inference

1. a) Note that $E[u_i^2|X] = \sigma_i^2$ implies $u_i^2 = \sigma_i^2 + v_i$ where $E[v_i|X] = 0$. Assuming OLS is consistent then $\hat{u}_i^2 \approx u_i^2$ for n large (do not need to be able to prove this). The null of no HC is $\eta_1 = 0$.

   Regression (2) can test σ_i that is linear in x_i (or can be well approximated by a linear function of x_i), namely for linear forms of heteroscedasticity(HC).

   Regression (3) is the regression used to form the White Test, noting that that all interaction terms in regressors are included since we have only a constant and x_i. The White Test has power against most alternative (no need to prove this). The null of no HC is $\beta_1 = \beta_2 = 0$. Both can be test by an F-test (or a t-test in (2) with one restriction) or by noting that under the null nR^2 has a chi-squared limit distribution with degrees of freedom equal to the number of restrictions under the null.

   To perform the F-test in (3)

   - Regress $\widehat{u}_i$ on a constant, x_i and also x_i^2. Under the null hypothesis that $E[u_i^2|X] = \sigma^2$ then $\beta_1 = \beta_2 = 0$ in (3).
   - The null hypothesis is that all slope coefficients are zero and hence the F-stat simplifies to $\frac{R^2/2}{(1-R^2)/(1000-3)}$.
   - Accept the null if $\frac{R^2/2}{(1-R^2)/997}$ is less than the 5% critical value comes from $F(2, 997)$.

   A similar procedure may be used in (2).

   b) We prefer (3). The R^2 has increased over ten fold relative to (2) which suggests that $\beta_2 \neq 0$ and that (2) will not pick up heteroscedasticity even though it seems to be present. Also the White Test has power against more alternatives and hence may have higher power against the unmown form of HC that may be present in u_i. We test $\beta_1 = \beta_2 = 0$ which is equivalent to an OLS that all slope coefficients are equal to zero. Hence the F-stat simplifies to $\frac{R^2/2}{(1-R^2)/(1000-3)}$ which in this case equals 60.7. The F-stat is asymptotically distributed as F(2,997) where the 5 % critical value is 3.02 and hence we reject the null at this significance level.

c) i. May prefer White Standard errors as strong evidence of heteroscedasticity from (b) and White Standard errors robust (for large enough sample sizes) to general forms of heteroscedasticity. Perform a t-test, $t = 0.389/1.362 = 0.286$ which is greater than 1.65 (the one sided 5% critical value of a $N(0,1)$.Hence we have no evidence to reject the null hypothesis.

ii. White s.e are robust to HC for a large enough sample size. NW standard errors robust to HC and SC (serial correlation). Since there is no evidence of SC we may prefer White Standards errors as the NW s.e are unlikely to remove any bias from the W estimates of the variance though will in likely provide a much less accurate estimate than W since it has a higher variance **(bias/variance tradeoff)**. NW preferable only if SC is present, the stronger the correlation the more NW likely removes a bias in the estimates of the s.e at the cost of an increase in the variance of our estimate of the s.e.

2. a) To show unbiasedness we need to establish that $E[u_i|X] = 0$ which (maintaining linearity and no perfect multicollinearity) implies $E[\hat{\beta}] = \beta_0$.

$$\begin{aligned} E[u_i|X] &= E[\eta_i x_{1i}|X] && \text{Plugging in } u_i = \eta_i x_{1i} \\ &= x_{1i}E[\eta_i|X] && \text{By properties of conditional expectation} \\ &= 0 && \text{As by assumption} E[\eta_i|X] = 0 \end{aligned}$$

This is crucial in showing that OLS is unbiased. We can now use the standard proof. We can show that $E[\hat{\beta}|X] = \beta_0$ which taking expectations on both sides implies $E[\hat{\beta}_T] = \beta_0$ [**Intuition:** If the mean of $\hat{\beta}_T$ is β_0 conditional X taking any value, then $\hat{\beta}_T$ has to have mean β_0 overall].

$$
\begin{aligned}
E[\hat{\beta}_T|X] &= E\left[\left(\sum_{t=1}^{T} x_t x_t'\right)^{-1}\left(\sum_{t=1}^{T} x_t y_t\right)\Big| X\right] && \text{Plug in formula for OLS estimator} \\
&= \beta_0 + E\left[\left(\sum_{t=1}^{T} x_t x_t'\right)^{-1}\left(\sum_{t=1}^{T} x_t u_t\right)\Big| X\right] && \text{Plugging in } y_t = x_t'\beta_0 + u_t \\
&= \beta_0 + \left(\sum_{t=1}^{T} x_t x_t'\right)^{-1}\sum_{t=1}^{T} x_t E\left[u_t|X\right] && \text{Conditioning on X so x is fixed} \\
&= \beta_0 \quad \text{As } E[u_t|X] = 0 \text{ for all t}
\end{aligned}
$$

b) In the following it is simpler to derive the result using the equivalent expression for the OLS estimator $\hat{\beta}_T = (X'X)^{-1}X'y$ noting that $X'X = \sum_{t=1}^{T} x_t x_t'$ and $X'y = \sum_{t=1}^{T} x_t y_t$.

$$
\begin{aligned}
Var(\hat{\beta}_T|X) &= E\left[\left(\hat{\beta}_T - E[\hat{\beta}_T|X]\right)\left(\hat{\beta}_T - E[\hat{\beta}_T|X]\right)'|X\right] && \text{By definition} \\
&= E\left[\left(\hat{\beta}_T - \beta_0\right)\left(\hat{\beta}_T - \beta_0\right)'|X\right] && \text{As } E[\hat{\beta}_T|X] = \beta_0 \\
&= E\left[(X'X)^{-1}X'uu'(X'X)^{-1}\right] && \text{As } \hat{\beta}_T - \beta_0 = (X'X)^{-1}X'u \\
&= (X'X)^{-1}X'E[uu'|X]X(X'X)^{-1} && \text{By properties of Conditional Expectation}
\end{aligned}
$$

Where $u = (\eta_{1i}x_{1i}, .., \eta_{ni}x_{1i})'$ Then $E[uu'|X]$ has $ij'th$ element where (i=j) is equal to $E[\eta_i^2 x_{1i}^2|X] = x_{1i}^2\sigma_\eta^2$ and when $i \neq j$ is $x_{1i}x_{1j}E[\eta_i\eta_j|X] = 0$ as $E[\eta_i\eta_j|X] = 0$ by assumption. Hence

$$X'E[uu'|X]X = \sigma^2\sum_{i=1}^{n} x_{1i}^2 x_i x_i'$$

as all off-diagonal terms are equal to zero.

Hence

$$\sigma^2(X'X)^{-1}\sum_{i=1}^{n} x_{1i}^2 x_i x_i'(X'X)^{-1}$$

And OLS is not efficient.

c) Use Weighted Least Squares. If we divide (5) by x_{1i} then,

$$y_i/x_{1i} = x_i/x_{1i}'\beta_0 + u_i/x_{1i} \tag{1}$$

Where $u_i/x_{1i} = \eta_i$ where $E[\eta_i^2|X] = \sigma_\eta^2$ and $E[\eta_i\eta_j|X] = 0$ for $i \neq j$ and hence η_i is a Spherical Error. Define $y_i^* = y_i/x_{1i}$ and $x_i^* = x_i/x_{1i}$ and $u_i^* = u_i/x_{1i} = \eta_i$ and $y^* = (y_1^*, .., y_n^*)'$, $X^* = (x_1^*, .., x_n^*)'$ then an estimate $\beta^* = (X^{*'}X^*)^{-1}X^{*'}y^*$ will be unbiased since the error term $E[\eta_i|X] = 0$ and since η_i is spherical then we can show,

$$Var(\hat{\beta}^*|X) = \sigma_\eta^2(X^{*'}X^*)^{-1} \tag{2}$$

and satisfies the transformed regression satisfies the Gauss Markov assumption and is BLUE, unlike the OLS estimator.

d) The WLS estimator $\hat{\beta}^*$ will not be consistent since now the incorrect weights are used. Now if $u_i = \eta_i x_{2i}$ hence $u_i/x_{1i} = \eta_i x_{2i}/x_{1i}$ and $E[u_i^2|X] = \sigma_\eta^2(x_{2i}/x_{1i})^2$ and hence the errors do not have constant variance and hence the Gauss Markov Theorem does not hold and OLS is not efficient.

3. a) We would use them if we had established that the error terms were autocorrelated. Using these in t-tests will, asymptotically, ensure that the resulting t-tests are normally distributed. However if the sample size is very small this may not deliver correct inference (i.e the tests may not have the correct size).

b) We can test for autocorrelation (of order 1). $H_0 : \eta_5 = 0$ (no autocorrelation of order 1) H_A : any $\eta_5 \neq 0$ (autocorrelation of up to order 1). Reject H_0 if χ^2 is larger than c.v. Test Stat $\chi^2 = TR^2 = 411 * 0.0182 = 7.48$ is larger than χ^2cv with 1 d.o.f (=3.84). Do reject H_0 of no autocorrelation.
Hence we got to use the NW se for a t-test.
$H_0 : \beta = 0.5; H_A : \beta \neq 0.5$.Dec Rule: Reject H_0 if $|t - test| > c.v.$ $t - test = (0.721 - 0.5)/1.482 = 0.149$. 2 sided cv for $\alpha = 0.05$ is 1.96. $|t - test| < 1.96$ hence we do not reject the null hypothesis.
$H_0 : \gamma \geq 0; H_A : \gamma < 0$. Dec Rule: Reject H_0 if $t - test < c.v.$ $t - test = (-1.581)/0.523 = -3.023$. 1 sided cv for $\alpha = 0.05$ is -1.645. $t - test < -1.645$ hence we do reject the null hypothesis.