3.(a) Consider the Linear Model

$$y_t = \beta_0 x_t + u_t \tag{1}$$

where x_t is scalar and (y_t, x_t, u_t) for $t = 1, ., T$ forms an i.i.d sequence. Define $X = (x_1, .., x_T)'$ where $E[u_t|X] = 0$, $E[u_t^2|X] = \sigma_t^2$ for all $t = 1, .., T$ and $E[u_t u_s|X] = 0$ for all $t \neq s$. Define $SSR_{wls} = \sum_{t=1}^{T} w_t u_t(\beta)$ where $u_t(\beta) = y_t - \beta x_t$ and w_t $(t = 1, .., T)$ are a set 'weights' that may be a function of X. Define $\hat{\beta}_{wls}$ as the value of β that minimises $\hat{\beta}_{wls}$ for a given set of weights w_t $(t = 1, .., T)$.

(i) Show that $\hat{\beta}_{wls} = (\sum_{t=1}^{T} w_t x_t^2)^{-1} \sum_{t=1}^{T} w_t x_t y_t$. **[4 marks]**

(ii) A researcher claims $\hat{\beta}_{wls}$ will be an unbiased estimator of β_0 if and only if $\sum_{t=1}^{T} w_t = 1$. Discuss with formal proof the validity of this assertion. **[4 marks]**

(ii) Demonstrate that $Var(\hat{\beta}_{wls}|X) = \frac{\sum_{t=1}^{T} w_t \sigma_t^2 x_t^2}{(\sum_{t=1}^{T} w_t x_t^2)^2}$. **[10 marks]**

3.(b) (i) A researcher considers the case where $x_t = 1$ for $t = 1, .., T$ and wishes to choose the weights w_t $(t = 1, .., T)$ such that $\sum_{t=1}^{T} w_t = 1$ to minimise $Var(\hat{\beta}_{wls}|X)$. Show that the solution to the problem is $w_t = \sigma_t^{-1}(\sum_{t=1}^{T} \sigma_t^{-1})^{-1}$ for $t = 1, .., T$. [*Hint: Setup the Lagrangean for this problem and show* $w_t = \sigma_t^{-1}(\sum_{t=1}^{T} \sigma_t^{-1})^{-1}$ *for* $t = 1, .., T$ *satisfies the Lagrangean Conditions.*] **[12 marks]**

(ii) Provide an intuitive explanation of why $w_t = \sigma_t^{-2}(\sum_{t=1}^{T} \sigma_t^{-2})^{-1}$ for $t = 1, .., T$ minimises the variance of $\hat{\beta}_{wls}$. **[3 marks]**

4.(a) Consider the Linear Model

$$y_t = \beta_0 x_t + u_t \tag{2}$$

$$x_t = \pi_0' z_t + \eta_t \tag{3}$$

where x_t is scalar and z_t is $m \times 1$ and $(y_t, x_t, z_t', u_t, \eta_t)'$ for $t = 1, .., T$ forms an i.i.d sequence. Define $Z = (z_1, .., z_T)'$ where $E[u_t z_t] = c$ where c is an $m \times 1$ vector, $E[z_t z_t'] = \Sigma_0$ for some $m \times m$ bounded matrix Σ_0, $E[u_t^2|Z] = \sigma_0^2$ for $t = 1, .., T$ and $E[u_t u_s|Z] = 0$ for $t \neq s$. A researcher forms an instrument $k_t = \alpha' z_t$ for some $\alpha \in \mathbb{R}^m$ to form the IV estimator $\hat{\beta}_{IV} = (\sum_{t=1}^{T} k_t x_t)^{-1}(\sum_{t=1}^{T} k_t y_t)$.

(i) Show that the IV estimator is identified when $\alpha' \Sigma_0 \pi_0 \neq 0$. **[3 marks]**

(ii) Derive the probability limit of $\hat{\beta}_{IV}$ as a function of $\beta_0, \pi_0, \Sigma_0, c, \alpha$ when $\alpha' \Sigma' \pi_0 \neq 0$. **[8 marks]**

(iii) A researcher argues that $\hat{\beta}_{IV}$ may consistently estimate β_0 when $\alpha' \Sigma' \pi_0 \neq 0$ even when $c \neq 0$. Using your answer in a(ii) discuss why this is the case. **[2 marks]**

(iv) Under the condition in a(iii) and $\alpha'\Sigma_0\pi_0 \neq 0$ ensuring $\hat{\beta}_{IV}$ is consistent, show that $\sqrt{T}(\hat{\beta}_{IV} - \beta_0) \xrightarrow{d} N(0, V)$ where $V = \sigma_0^2 \frac{\alpha'\Sigma_0\alpha}{(\alpha'\Sigma_0\pi_0)^2}$. [**12 marks**]

4.(b) If $\alpha'\Sigma_0\pi_0$ is close to (but not equal to) zero, briefly discuss two problems in using $\hat{\beta}_{IV}$ to form inference on β_0. [**4 marks**]

4.(c) If $E[x_t u_t] = 0$ a researcher argues that OLS is always preferable to IV to form inference on β_0. Discuss the validity of this statement. [**4 marks**]