3.(a) Consider the Linear Model

$$y_t = \beta_0 x_t + u_t \tag{1}$$

(i) Show that $\hat{\beta}_{wls} = (\sum_{t=1}^{T} w_t x_t^2)^{-1} \sum_{t=1}^{T} w_t x_t y_t$. **[4 marks]**

Differentiate $SSR_{wls} = \sum_{t=1}^{T} w_t u_t(\beta)$ w.r.t β and set equal to zero to find $\hat{\beta}_{wls} = (\sum_{t=1}^{T} w_t x_t^2)^{-1} \sum_{t=1}^{T} w_t x_t y_t$..

(ii) A researcher claims $\hat{\beta}_{wls}$ will be an unbiased estimator of β_0 if and only if $\sum_{t=1}^{T} w_t = 1$. Discuss with formal proof the validity of this assertion. **[4 marks]**

Statement is incorrect, will be unbiased always so long as $\sum_{t=1}^{T} w_t x_t^2 > 0$ so the estimator exists. A derivaton of $E[\hat{\beta}_{wls}]$ is needed to show this formally.

(iii) Demonstrate that $Var(\hat{\beta}_{wls}|X) = \frac{\sum_{t=1}^{T} w_t \sigma_t^2 x_t^2}{\left(\sum_{t=1}^{T} w_t x_t^2\right)^2}$. **[10 marks]**

First show that $E[\hat{\beta}_{wls}|X] = \beta_0$ then notes

$Var(\hat{\beta}_{wls}|X) = E[\left((\sum_{t=1}^{T} w_t x_t^2)^{-1} \sum_{t=1}^{T} w_t x_t u_t\right)^2] = (\sum_{t=1}^{T} w_t x_t^2)^{-2} \sum_{t=1}^{T} w_t^2 x_t E[u_t^2|X]$

by no s.c assumption then note that $E[u_t^2|X] = \sigma_t^2$ giving the result. Must state all steps and report assumptions used.

3.(b) (i) A researcher considers the case where $x_t = 1$ for $t = 1, .., T$ and wishes to choose the weights w_t $(t = 1, .., T)$ such that $\sum_{t=1}^{T} w_t = 1$ to minimise $Var(\hat{\beta}_{wls}|X)$. Show that the solution to the problem is $w_t = \sigma_t^{-1}(\sum_{t=1}^{T} \sigma_t^{-1})^{-1}$ for $t = 1, .., T$. [*Hint: Setup the Lagrangean for this problem and show $w_t = \sigma_t^{-1}(\sum_{t=1}^{T} \sigma_t^{-1})^{-1}$ for $t = 1, .., T$ satisfies the Lagrangean Conditions.*] **[12 marks]**

Setup the Lagrangean (noting that $x_t = 1$) so that $Var(\hat{\beta}_{wls}|X) \sum_{t=1}^{T} w_t \sigma_t^2$ since $\sum_{t=1}^{T} w_t = 1$ and

$\mathcal{L} = \sum_{t=1}^{T} w_t^2 \sigma_t^2 - \lambda(\sum_{t=1}^{T} w_t - 1)$

diff w.r.t w_t to find

$2w_t\sigma_t^2 - \lambda = 0$ and $w_t = \lambda/2\sigma_t^2$ where $\sum w_t = 1$ so $1 = \lambda/2 \sum_{t=1}^{T} \sigma_t^{-2}$ and

$\lambda = 2\left(\sum_{t=1}^{T} \sigma_t^{-2}\right)$ which we can sub back in to the above to show

$w_t = \sigma_t^{-2}\left(\sum_{t=1}^{T} \sigma_t^{-2}\right)$.

(ii) Provide an intuitive explanation of why $w_t = \sigma_t^{-2}(\sum_{t=1}^{T} \sigma_t^{-1})^{-2}$ for $t = 1, .., T$ minimises the variance of $\hat{\beta}_{wls}$. **[3 marks]**

The estimator putting more weight to that part of the sample with the smallest variation in u_t which is 'more informative' about β_0

4.(a) (i) Show that the IV estimator is identified when $\alpha'\Sigma_0\pi_0 \neq 0$ **[3 marks]**
$E[k_t x_t] = \alpha' E[z_t' x_t] = \alpha'\Sigma_0\pi_0$ which is full rank when $\alpha'\Sigma_0\pi_0 \neq 0$.

(ii) Derive the probability limit of $\hat{\beta}_{IV}$ as a function of $\beta_0, \pi_0, \Sigma_0 c, \alpha$ when $\alpha'\Sigma'\pi_0 \neq 0$. **[8 marks]** $\hat{\beta}_{IV} = \beta_0 + (1/T\sum_{t=1}^{T} k_t x_t)^{-1}(1/T\sum_{t=1}^{T} k_t u_t)$. where $1/T\sum_{t=1}^{T} k_t x_t \xrightarrow{p} E[k_t x_t] = \alpha' E[z_t' x_t] = \alpha'\Sigma_0\pi_0$ by WLLN under i.i.d

When $\alpha'\Sigma_0\pi_0 \neq 0$ then $1/T\sum_{t=1}^{T} k_t x_t \xrightarrow{p} (\alpha'\Sigma_0\pi_0)^{-1}$ by CMT. $1/T\sum_{t=1}^{T} k_t u_t \xrightarrow{p} E[k_t u_t] = \alpha' c$
By slutsky $\hat{\beta}_{IV} \xrightarrow{p} \alpha' c/\alpha'\Sigma'\pi_0$

(iii) A researcher argues that $\hat{\beta}_{IV}$ may consistently estimate β_0 when $\alpha'\Sigma'\pi_0 \neq 0$ even when $c \neq 0$. Using your answer in a(ii) discuss why this is the case. **[2 marks]**
Consistent when $\alpha' c = 0$.

(iv) Under the condition in a(ii) show that $\sqrt{T}(\hat{\beta}_{IV} - \beta_0) \xrightarrow{d} N(0, V)$ where $V = \sigma_0^2 \frac{\alpha'\Sigma_0\alpha}{(\alpha'\Sigma_0\pi_0)^2}$. **[12 marks]**
$\sqrt{T}(\hat{\beta}_{IV} - \beta_0) = (1/T\sum_{t=1}^{T} k_t x_t)^{-1}(1/\sqrt{T}\sum_{t=1}^{T} u_t k_t$ then by i.i.d the WLLN and CMT imply $(1/T\sum_{t=1}^{T} k_t x_t)^{-1} \xrightarrow{p} \alpha'\Sigma_0\pi_0)^{-1}$ since $\alpha'\Sigma_0\pi_0) \neq 0$

By CLT noting that $E[u_t k_t] = 0$ when $\alpha' c = 0$ from a(ii) where $Var(1/\sqrt{T}\sum_{t=1}^{T} u_t k_t) = \sigma_0^2\alpha'\Sigma_0\alpha$ then $1/\sqrt{T}\sum_{t=1}^{T} u_t k_t \xrightarrow{d} N(0, \sigma_0^2\alpha'\Sigma_0\alpha)$

4.(b) If $\alpha'\pi_0$ is close to (but not equal to) zero, briefly discuss two problems in using $\hat{\beta}_{IV}$ to form inference on β_0. **[4 marks]**

IV has higher variance (2marks) Distribution of IV estimator less well approximated by normal distribution (2marks)

4.(c) If $E[x_t u_t] = 0$ a researcher argues that it OLS is always preferable to IV to form inference on β_0. Discuss the validity of this statement. **[4 marks]**

Statement valid, OLS is conistent and will have lower variance since $Var(x_t) > Var(k_t)$.